MathEnriched

INDIVIDUALIZED NON-ALGEBRA UNITS

EASY TO VERY DIFFICULT

FOR GRADE 6 TO ADULT

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A Personal Note

This book, written over a period of fifty years, features enriched materials for seventh and eighth grade math students that can be used in class or at home.

After teaching high school math for a few idyllic years on the Island of Martha's Vineyard, it was startling to find little difference in the innate reasoning of grade 7 and 8 math students in Concord, Massachusetts. It was clear that these eager students needed something better. A sense of the variety and depth of the resulting material may be had by browsing the *Table of Contents, Annotative*. It is my hope that students will find many rich opportunities in these units and in the 180 "Today's Twisters", pages 411-465.

Any math teacher who does not belong to The National Council of Teachers of Mathematics may go to *nctm.org* and choose the appropriate level to join. Their middle school journal is superb in the development of specific math ideas and their learning for students and teachers.

Jennie Brown, without you I might still be looking at computers as an unnecessary complication in the life of a retired math teacher. Your persistent persuasive powers and encouragement have been helpful right from the beginning. Your editing and formatting skills have improved this material in substance and presentation. I am grateful.

Thank you Betsy Green for your help. You are the granddaughter who has amazed me with your persistence ever since you were a tiny tot finding the plastic cylinder that belonged next in the stack. No complaints, no exasperation, just work until it was done or simply not worth more attention. What wonderful portent!

Thank you to Stacy Dickie for helping to get some of the early, difficult "knots" untied. A mind with rare analytical gifts.

Thanks also to Spencer Davis who started me onto this project and was cheerfully available for making suggestions along the way. The result is quite different from what either of us thought. Apologies for not having this ready in time for your children.

And thanks to Keith Herndon, a very welcome help in questions mathematical.

To be Read by the Student, Parent, Teacher, Tutor

These supplementary units are intended chiefly for voluntary self-study by students in grades 7 and 8, exceptional students in grade 6, and interested people of any age. Support work is provided in Units 1 - 11 for those who need reminders from pre-algebra work. These early units also give some novel content and persuasive sequences of exercises.

A capable and caring teacher is the focus for learning. The teacher's behavior, body language and attitude communicate truly vital information for student growth. These self-instruction math units are intended as one of the teacher's tools.

Answers are supplied and thereby require from the student a special honesty with self. This means patience and perseverance in dealing with incomplete understanding. Quizzes are provided for most units. Motivated and capable students may work well on their own but even for them, sustained progress often depends upon access to a willing peer, parent, tutor or teacher.

An important purpose of this enriched material is that students come to understand some mature mathematical ideas. As in the following, expect to think, understand and rarely memorize procedures. Infer the meaning of [ ] and confirm the answers mentally. Units 43, 44 and 46 have rich expansions of these compelling requests for estimating and seeing relations.

The greatest integer in a number: $[\,3.7\,] = 3$, $[\sqrt{3.5}\,] = 1$, $[\,.99^{99}] = 0$, $\left[\frac{567}{568} + \frac{1}{569}\right] = 0$ $\left[\frac{567}{568} + \frac{568}{567}\right] = 2$, and $\left[\sqrt{\sqrt{\sqrt{2}}}\right]$ 1; is an unsurpassed technique for estimating and revealing relationships in fractions, decimals, repeating decimals and more. The basic ideas and development of the greatest integer function as a teaching tool, what he called "Square Brackets", were conceived by Professor David Page, University of Illinois in Urbana, himself a stellar example of what a teacher can be.

A level of thinking for such elevated topics as *Cantor's Transfinite Numbers*, Unit 36 (The Abyss), needs certain supporting knowledge, as do many of the units. There is enough support and refresher work, about 20% of content, to prepare the student for novel material and to deepen math background. Some units and problems are difficult. Don't give up easily. Difficulty ranges from no asterisk: ( ) Easy; (*) Not so easy; (**) difficult; (***) very difficult, for those willing to engage in very challenging work (See Table of Contents).

To be Read by the Student, Parent, Teacher, Tutor

MathEnriched units are suggested, but not limited, for use in these ways:

1) Voluntary participation is recommended. The somewhat gifted student in grade 6, working individually, might need all of the support units 1 - 10. An older student may read through familiar pages checking whether or not it is "support work" for her/him.
2) The seventh or eighth grade student who can handle extra-class challenge can gain many new insights and knowledge from these units working individually. A public record of individual progress in the units is a motivator of interest to many.
3) An above average seventh or eighth grade math class could use the material as basic work with supplementing from a traditional text. A suggested list of topics from eighth grade *not* covered here, or only partially covered, is given in Appendix I.
4) Some units could be used class wide to supplement or replace the usual class work.
5) Any interested high school student or adult.
6) A source for students and teachers to select units for interesting enriched work.
7) Catch the languishing potential in the minds of those with unknown or reluctant talent.

A loose leaf notebook for these math units is needed by anyone undertaking significant participation. It will become the student's only source for much of this material.

A calculator is needed. The calculator needs to be "scientific" and have the key $\boxed{y^x}$ or $\boxed{x^n}$ for raising to a power. A *computer* usually has a calculator with the expression x^y, or simply ^ as it is on some hand-held calculators, meaning x^y. A calculator separate from the computer is best. A ten-digit readout is suggested. **Important:** In some cases in the *MathEnriched* units it is specifically intended and conveyed that a calculator *not* be used because this will spoil the learning for which the exercises are designed.

The final fifty-one pages comprise a program of 180 daily twisters arranged four to a page in quadrants for ease of printing, cutting and daily display for student access. A public chart of achievement attracts considerable interest. Voluntary participation is suggested.

I hope that this material brings learning which is gratifying to student and teacher.

Feedback of any kind is welcome.

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TABLE OF CONTENTS, Annotative

Difficulty: ( ) Easy; (*) Not so easy; (**) difficult; (***) very difficult.

NUMBER GIANTS OF THE UNIVERSE

Unit 1

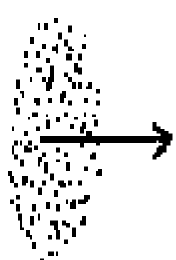
If you have ever looked at the sky on a sparkling clear night you have probably noticed a faint white band stretching across the heavens. Perhaps you already know that this is called the Milky Way and that it is made up of many stars so far away that they appear to us as a dim milky band.

A strange thing about the Milky Way is that our sun with its planets (including us, of course) is also a member of the Milky Way. This must mean that we on earth are in that very same milky band that we see so far away. This seems hard to believe. How could it be?

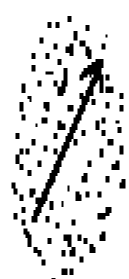
Actually, the Milky Way is not shaped like a band at all. If you could stand off in space and look at the whole thing at once, it would look quite a lot like masses of stars in the form of a spinning wheel, although you would not notice any spinning motion. It is quite round, but not very thick, just as you would expect a wheel to be.

Our earth is located in the wheel part way in from the outer rim. If you were standing on earth looking directly out from the wheel as in A, below, you would not see very many stars even on a clear night, probably only a few thousand. But, if you look along the wheel as in B, you can see that you would be looking through a tremendous sea of stars and that all except a few close ones would blend into a pale white band.

A



B



You read a minute ago that the Milky Way is not very thick. To be a little more exact, it is 300,000,000,000,000,000 miles thick. If you bothered to count the groups of zeros and know how to name some very large numbers, you might have discovered that the figure is three hundred quadrillion miles. That’s the same as having a billion miles 300 million times.

NUMBER GIANTS OF THE UNIVERSE

Unit 1

It begins to look as though people who talk and write about the Milky Way ought to have some way of writing large numbers without tiring out their hand or seeing spots in front of their eyes from writing so many zeros. This is especially true when we realize that there are countless other "wheels" in our universe and the tremendous distances which separate them make our wheel look like a dot on the chalkboard. A more scientific name for these wheels is "galaxy."

Of course, there is a way of shortening our tiresome way of writing very large numbers. Read the rest of this story of giant numbers carefully and do the exercises as you go along. You will soon be able to write the number of miles across our galaxy--1,000,000,000,000,000,000 miles--in a space no bigger than this ____ and with room left over, too! That's one quintillion miles.

First, look at the figures below and finish out the answers to the bottom.

2 x 2 = 4
2 x 2 x 2 = 8
2 x 2 x 2 x 2 = 16
2 x 2 x 2 x 2 x 2 = ___
2 x 2 x 2 x 2 x 2 x 2 = ___
2 x 2 x 2 x 2 x 2 x 2 x 2 = ____

3 x 3 = 9
3 x 3 x 3 = 27
3 x 3 x 3 x 3 = ____
3 x 3 x 3 x 3 x 3 = _____
3 x 3 x 3 x 3 x 3 x 3 = ______

Perhaps you realized that the answer to the fifth bunch of twos all multiplied together was 64 just by glancing at the previous answers. And the final answer was 64 x 2, or 128. Likewise in the 3's, the first answer you were to supply was 81, then 243 and 729. If you did it this way you should pause and give yourself a pat on the back.

Let's look at 2 x 2 x 2 x 2 x 2 a little more closely. There ought to be a shorter way of indicating that all those twos are to be multiplied together. Saying five twos obviously wouldn't work since five twos equals ten. So, mathematicians agreed a long time ago that any number, like 2, written with a little 5 above and to the right of it means that the two should be written down five times and all of them multiplied together. It would look like this: 2^5 equals 32. If you read 2^5 out loud you would say "Two to the fifth power."

3^4 would be read "three to the fourth power" and would mean 3 x 3 x 3 x 3 = ______.

You work it out.

5^2 is usually read "five squared" not "five to the second power", and 6^3 is usually read "six cubed" not "six to the third power". Relax, those are the only exceptions.

NUMBER GIANTS OF THE UNIVERSE — Unit 1

Do exercises 1 - 15 now. Remember, 2^3 does <u>not</u> mean 2 x 3.

Answers are on page 6. Check them when you finish 1 - 15.

1) 4^3 = _____ 2) 5^2 = _____ 3) 6^3 = ______ 4) 2^3 = ____

5) Two to the fourth power = _____ (work it out) 6) Seven squared = _____

7) Three cubed = _______ 8) 3^4 = _______

9) 5^3 = _________ 10) 8^2 = ______ 11) 3^3 = __________

12) Does 8^2 equal 2^8 ? (yes/no) __________ 13) Does 3^2 equal 2^3 ? __________

14) Does 2^4 equal 4^2 ? ___________ 15) Does 2^5 equal 5^2 ? __________

Now let's get back to the Milky Way. What does all this cube and square business have to do with 1,000,000,000,000,000,000 anyway? If you look at the next array of figures and finish it to the bottom you might get an idea or two.

$$10 \times 10 = 10^2 = 100$$
$$10 \times 10 \times 10 = 10^3 = 1000$$
$$10 \times 10 \times 10 \times 10 = 10^4 = 10{,}000$$
$$10 \times 10 \times 10 \times 10 \times 10 = 10^5 = 100{,}000$$
$$10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6 = \_\_\_\_\_\_\_\_\_\_\_\_$$
$$10^7 = \_\_\_\_\_\_\_\_\_\_\_\_\_$$
$$10^8 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Perhaps you noticed a pattern connected with the little numbers to the right of the tens, 10^4 = one with four zeros after it. 10^8, if you worked it out correctly, should equal one with eight zeros after it, or one hundred million. Look back to the column to get used to the idea if you want to. By the way, those little numbers which tell you how many times to write the number and then multiply are called <u>exponents.</u>

Let's turn the idea around backwards. Since the exponent of ten is the same as the number of zeros, all we have to do when we see a number like 100,000 is to count the number of zeros and use this number of zeros as our exponent (100,000 = 10^5).

How would you write 10,000,000? _____

NUMBER GIANTS OF THE UNIVERSE

Unit 1

Now you are ready to write the distance across the Milky Way --1,000,000,000,000,000,000 miles -- in a space no larger than this: _____. Do it.

Some very difficult problems could be given in a very small space.

For example, 2^{50} = ? (Don't do it!!)

The answer is sixteen digits long, more than 1 quadrillion. Perhaps that's a reason why such a problem is called "2 to the fiftieth *power*".

Here are some that are not that much work but which should be done carefully.

1) 3^7 = __________ 2) 2^7 = __________ 3) 4^5 = ___________

4) 9^4 = __________ 5) 5^4 = __________ 6) 5^5 = ___________

Now let's recall a fact that you might have forgotten. In order to multiply a whole number by ten, all you have to do is add a zero to that number (10 x 45 = 450). To multiply by one hundred you add two zeros; by one thousand you add three zeros; and so on.

15 x 1000 = 15,000

15,000 = 15 x 1,000 = 15 x 10^3

Since 60,000 = 6 x 10,000, then 60,000 = 6 x 10^4

Since 12,500,000 = 125 x 100,000, then 12,500,000 = 125 x 10^5

Use the patterns on the previous page for examples 7 & 8.

7) Since 6,000,000 = 6 x 1,000,000, then 6,000,000 = 6 x ____

8) 280,000,000 = 28 x _____

Write the answers for exercises 9, 10 and 13 in this form**:** 28 x 10^7.

9) 473,000 = ______________ 10) 14,000,000,000,000 = _________

11) 123 x 10^4 = ______________ 12) 3 x 10^3 = ___________________

13) 62,400,000,000 = ___________ 14) 63 x 10^{10} = __________________________

15) Now write the thickness of the Milky Way using exponents. ________ .

(See page 2, paragraph 1 of this unit)

NUMBER GIANTS OF THE UNIVERSE Unit 1

*16) What would be the thickness if it were twice as thick? Give answer using 10 with an exponent as you did in example 15. ____________________________

17a) 450,000,000,000 = _______ x $10^{\text{-------}}$

*b) 37000 . . . (total of 463 zeros) . . . 000 = 37 x $10^{\text{-----}}$

*18a) = $2^{\left(3^2\right)}$ ______ b) $\left(3^2\right)^3 =$ _____ c) $3^{\left(2^3\right)} =$ __________ d) $\left(2^3\right)^2 =$ ________

*19) True or false: a) $2^{\left(3^2\right)} = \left(2^3\right)^2$ _______ b) $3^{\left(45^{100}\right)} = \left(3^{45}\right)^{100}$ _______

20a) 10^{10} = 1 followed by how many zeros? ____

*b) $10^{\left(10^{10}\right)}$ = 1 followed by how many zeros?___________

Try using no exponents in your answer.____

Answers on next page.

NUMBER GIANTS OF THE UNIVERSE — Unit 1

Answers

Page 3:

1) 64 **2)** 25 **3)** 216 **4)** 8 **5)** 16 **6)** 49 **7)** 27
8) 81 **9)** 125 **10)** 64 **11)** 27 **12)** no **13)** no **14)** yes
15) no

Page 4:

10^6 = 1,000,000 10^7 = 10,000,000 10^8 = 100,000,000

10,000,000 = 10^7

The distance across the Milky Way is written 10^{18} miles.

1) 2187 **2)** 128 **3)** 1024 **4)** 6561 **5)** 625 **6)** 3125

Page 5:

7) 10^6 **8)** 10^7 **9)** 473 x 10^3 **10)** 14 x 10^{12} **11)** 1,230,000

12) 3,000 **13)** 624 x 10^8 **14)** 630,000,000,000

15) The thickness of the Milky Way is 3 x 10^{17} miles

16) 6 x 10^{17} Note: Likely wrong answers are 3 x 10^{34} or 3 x 20^{17}. Either of these is many, many times the value of 3 x 10^{17}.

17a) 45 x 10^{10} *** b)** 37 x 10^{463}

***18a)** $2^9 = 512$ **b)** $9^3 = 729$ **c)** $3^8 = 6561$ **d)** $8^2 = 64$

19a) False. You can try it out: $2^{(3^2)} = 2^9 = 512$

b) False. Same **pattern** as a). You don't need to try it out to know it is wrong.

20a) 10 ***b)** $10^{\left(10^{10}\right)} = 10^{10,000,000,000}$, or (1 followed by 10,000,000,000 zeros, or 10 billion zeros).

NUMBER GIANTS OF THE UNIVERSE 2 *Unit 2

Recall from "Number Giants of the Universe": $6^3 = 6 \times 6 \times 6 = 216$

$10^4 = 10 \times 10 \times 10 \times 10 = 10{,}000$ $3 \times 10^3 = 3{,}000$ $280{,}000 = 28 \times 10^4$

1) 10^2 = _____ 2) 5^3 = ____ 3) 2^4 = ____

4) 10^3 = _____ 5) 5×10^3 = _________ 6) $60{,}000 = 6 \times 10^{\square}$

Note: $\left(\frac{1}{2}\right)^3 = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$, not $\frac{1}{6}$

7) $\left(\frac{1}{4}\right)^2 =$ ______ (not $\frac{1}{8}$) 8) $\left(\frac{3}{5}\right)^2 =$ _________ 9) $\left(\frac{2}{3}\right)^3 =$ _________

10) Show any work on this paper (if any): $\left(\frac{2}{3}\right)^2 \times \left(\frac{3}{2}\right)^3 =$_______ 11) 123×10^4 = _______

12) 237,000,000 = _______ x _______ (Use 10 with an exponent.)

13) 1,700,000,000,000,000,000,000 = ____x ____

<u>When decimals appear in a number, the rule changes</u>. In both columns below, each number is 10 times the number above it. See if you agree.

12.345		27.789
123.45		277.89
1234.5	Point anchored, number moves.	2778.9
12345.		27789.
123450.	Point moves, number anchored	277890.
1234500.		2778900.

For exercise 15, go <u>down</u> <u>two</u> steps in either of the two columns above left.

14) 12.3456 x 10 = __________ 15) 12.345×10^2 = _____________

16) 277.89 x 10 = __________ 17) 27.789×10^3 = _____________

18) 277.89×10^4 = ______________ (Add just enough zeroes to move 4 places.)

19) 123.45×10^2 = ____________ 20) 12.345×10^5 = __________________

NUMBER GIANTS OF THE UNIVERSE 2 *Unit 2

All of these problems can (and should) be done mentally.

21) $1234.5 \div 10 =$ __________ (Go up one step on page 1 to divide by ten.)

22) $2779.9 \div 10^2 =$ __________ 23) $1234.5 \div 10^{\square} = 12.345$ 24) $28.8 \div 10^2 =$ ________

25) $2.95 \div 10^2 = .0295$, so $45.67 \div 10^4 =$ _________ 26) $2.6 \div 10^{\square} = .0026$

27) $972 \div 1{,}000 =$ _____ 28) $.003678 \times 10^{\square} = 367.8$

29) $100{,}000{,}000{,}000 \div 10^{10} =$ ____ *30) $10^{12} \div 10^{10} =$ _______

Exercise 10 was $\left(\frac{2}{3}\right)^2 \times \left(\frac{3}{2}\right)^3 = ?$ and had the answer $\frac{3}{2}$, or $1\frac{1}{2}$. It is not a coincidence that the answer is a number in the problem. Note that: $\frac{2}{3} \times \frac{3}{2} = \frac{6}{6}$ or 1

31a) $\frac{2}{3} \times \frac{3}{2} =$ b) $\left(\frac{2}{3} \times \frac{3}{2}\right) \times \left(\frac{2}{3} \times \frac{3}{2}\right) =$ c) $\frac{673}{76} \times \frac{76}{673} =$

d) $\left(\frac{2}{3} \times \frac{3}{2}\right) \times \frac{3}{2} =$ e) $\left(\frac{2}{5}\right)^2 \times \left(\frac{5}{2}\right)^3 =$ f) $\left(\frac{2}{3}\right)^3 \times \left(\frac{3}{2}\right)^2 =$

g) $\left(\frac{2}{3}\right)^4 \times \left(\frac{3}{2}\right)^2 =$ h) $\left(\frac{2}{3}\right)^5 \times \left(\frac{3}{2}\right)^2 =$ i) $\left(\frac{5}{7}\right)^{30} \times \left(\frac{7}{5}\right)^{31} =$

*32) $\left(\frac{3}{4}\right)^{90} \times \left(\frac{4}{3}\right)^{89} =$ *33) $\left(\frac{9}{7}\right)^{2000} \times \left(\frac{7}{9}\right)^{1998} =$

34) $\left(\frac{3}{4}\right)^{122} \times \left(\frac{4}{3}\right)^{120} =$ *35) $\left(\frac{2}{3}\right)^{650} \times \left(\frac{3}{2}\right)^{653} =$

36) $\left(\frac{61}{89}\right) \times \left(\frac{89}{\square}\right) = 1$ 37) $\left(\frac{7}{5}\right)^{(\ )} \times \left(\frac{5}{7}\right)^{27} = \frac{49}{25}$

*38 $\left(\frac{8}{5}\right)^{(\ )} \times \left(\frac{5}{8}\right)^{20} = 1\frac{3}{5}$. 39) $\left(\frac{17}{15}\right)^{56} \times \left(\frac{(\ )}{17}\right)^{58} = \frac{225}{289}$

*40 $\left(\frac{7}{11}\right)^{8} \times \left(\frac{11}{7}\right)^{(\ )} = 1\frac{4}{7}$

NUMBER GIANTS OF THE UNIVERSE 2

***Unit 2**

Answers

1) 100 **2)** 125 **3)** 16 **4)** 1,000 **5)** 5,000 **6)** ^{4}

7) $\frac{1}{16}$ **8)** $\frac{9}{25}$ **9)** $\frac{8}{27}$ **10)** $\frac{3}{2}$ or 1 1/2 **11)** 1,230,000

12) 237×10^{6} **13)** 17×10^{20} **14)** 123.456 **15)** 1,234.5

16) 2,778.9 **17)** 27,789 **18)** 2,778,900 **19)** 12,345

20) 1,234,500 **21)** 123.45 **22)** 27.799 **23)** ^{2}

24) 0.288 **25)** 0.004567 **26)** ^{3} **27)** 0.972

28) ^{5} **29)** 10 **30)** 100 or 10^{2}

31a) 1 **b)** 1 **c)** 1 **d)** 3/2 or $1\frac{1}{2}$ **e)** 5/2 or 2 1/2 **f)** 2/3

***g)** $(2/3)^{2}$ or 4/9 ***h)** $(2/3)^{3}$ or 8/27 **i)** 7/5 or 1 2/5

***32)** 3/4 ***33)** 81/49 or 1 32/49 **34)** 9/16 ***35)** 27/8 or 3 3/8

36) 61 **37)** ^{29} ***38)** ^{21} ***39)** 15 **40)** ^{9}

BING, A HUMAN BINARY COUNTER

Unit 3

Important Note: Do this whole paper before trying Bing with anyone else.

A computer operates on a yes or no basis. A switch is either open or closed. Electric current flows through a particular circuit or it does not. It is mainly the combinations of these switches which allow the logical and mathematical operations to be done for us by a computer. We can see this action in simple form in the game of Bing. You will need three or four people besides yourself to play this game. Even young children and adults can learn to play. Any extra people present can act as observers to catch mistakes which almost certainly will be made. Larger numbers of people can be used by giving turns to small groups.

The instructions you give will be simple and clear but must be followed exactly, and only upon receiving a signal. Have one person stand facing the group as a demonstrator and explain that there are only two positions which the person may take. One position is with arms at the sides (switch is off). The other position is with one arm raised (switch is on).

It doesn't matter which arm is raised but it should be held aloft such that there is no doubt in anyone's mind whether the arm is up or down. Fatigue, doubt or inattention can often cause an arm to flounder limply, neither up nor down. No limp floundering is permitted.

Explain that the signal for this one person is the clearly spoken word "Bing". Upon hearing "Bing", this person (switch) immediately raises an arm if both arms are down, but lowers the arm if an arm is up. Have her/him practice several times in response to the spoken "Bing", making sure to repeat the word slowly and deliberately, giving time for reaction. Have this person sit down and another person try being a switch until everyone understands clearly.

Now have a second person stand at the first person's right hand side (left side when viewed from the audience) and explain her/his task. This task is the same but the signal is not "Bing". The second person's signal will be the lowering of the first person's arm (not the raising). Watch for this as you have them practice together. You need to know what to expect, but they should not execute early. Each switch should wait for his/her instruction. After all, switches can't think. They never anticipate!

BING, A HUMAN BINARY COUNTER

Unit 3

When the first two people are well trained, have a third person stand at the second person's right side with the instruction that his/her signal to change position will be the lowering of the second person's arm. "Put arm up if down, and put arm down if up" is still the general instruction for a person to move. Have them practice while you call out "Bing". Watch for the common error of anticipating.

The chart at the right shows the positions of the human switches after the number of bings shown. 0 is for no hand up and 1 is for a hand up. Fill in the rest of the chart by deciding what the positions will be as you go down the chart to seven bings. You should end with 111 for seven bings. If not, trace back to see where the switching went astray.

C	B	A	Bing
0	0	0	0
0	0	1	1
0	1	0	2
0	1	1	3
			4
			5
			6
			7

Be sure that you change B's position only when A's position has gone from **up to down** (1 to 0). Change C's position only when B's arm **goes down**.

Now that you have filled in the chart to the bottom, this binary counter made of human switches has reached a point where a chain reaction has been set up. We need more switches. .

Answers to Bings

4, 5, 6, 7 are 100, 101, 110, 111

BING, A HUMAN BINARY COUNTER — Unit 3

F	E	D	C	B	A	
0	0	0	1	1	0	6
0	0	0	1	1	1	7
						8
						9
						10
						11
						12
						13
0	0	1	1	1	0	14
						15
						16
						17
0	1	0	0	1	0	18
0	1	1	1	0	1	29
						30
						31
						32
						33
						34

These first two rows (bings 6 and 7) have been carried over from the first page of this unit, and columns D, E, and F, have been added.

The columns on this page are to record the bing responses down to bing 35. The gap indicates bings 19 - 28 which we will not record. The filled in answers are to help you stay on track. Record the bings now, before you instruct the group, so you will be somewhat of an expert. Proceed consecutively by rows.

You could "set the switches" as in bing 14 or bing 29, by telling everyone whether to have an arm up or down, so participants may get to higher numbers or to interesting "chain reaction" locations more quickly.

Notice that column A changes after every row.

1) Column B changes after every ____ rows.

2) Column C changes after every ____ rows.

3) Column D changes after every ____ rows.

4) Column E? ____ rows.

5) Column F? ____ rows.

*6) If we had a column G, how many bings would it take from the very beginning for Column G to change from 0 to 1 ? _____

*7) If your answer to Exercise 6 was correct (See next page), Column G would be said to change at the sixty-fourth bing, at row sixty-four in other words. Write the zeros and ones which would appear in row 63. Don't bother drawing boxes. Squeeze your digits below bing 35.

*8) What are the next three sets of bing positions following 1011101?

____________________, _______________________, _______________________

BING, A HUMAN BINARY COUNTER Unit 3

F	E	D	C	B	A	
0	0	0	1	1	0	6
0	0	0	1	1	1	7
0	0	1	0	0	0	8
0	0	1	0	0	1	9
0	0	1	0	1	0	10
0	0	1	0	1	1	11
0	0	1	1	0	0	12
0	0	1	1	0	1	13
0	0	1	1	1	0	14
0	0	1	1	1	1	15
0	1	0	0	0	0	16
0	1	0	0	0	1	17
0	1	0	0	1	0	18

<u>Special notes</u>:

You could make cards with a large 1 written on the first, 2 on the second, 4 on the third, 8 on the fourth, 16 on the fifth and 32 on the sixth. Have the first person (the "A" column) hold the 1 card, the next person the 2 card, etc. They should be clearly displayed in the hand that is raised.

If each person does his/her job**:** the *sum* of the numbers on the cards held up should agree with the total number of "Bings" uttered. Of course, your chart should do the same.

Important:

0	1	1	1	0	1	29
0	1	1	1	1	0	30
0	1	1	1	1	1	31
1	0	0	0	0	0	32
1	0	0	0	0	1	33
1	0	0	0	1	0	34
1	0	0	0	1	1	35

A good variation would be to have the "binger" say "One, two, three, four," etc., instead of "Bing, bing, bing", etc. Everyone can then see that the human computer's card total is actually counting.

Answers 1 – 8

1) 2 **2)** 4 **3)** 8 **4)** 16 **5)** 32 **6)** 64

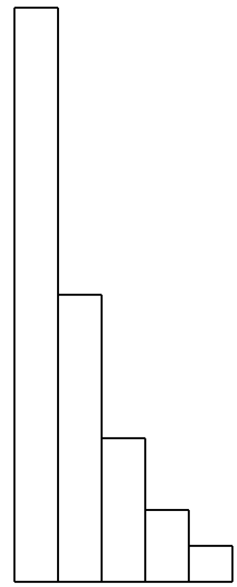
7) The zeros and ones in row 63 would be 0111111, or simply 111111...

8) 1011110_2, 1011111_2, 1100000_2

POWERS OF 2 AND THE MAGIC NUMBER CHART *Unit 4

Use scientific calculator where needed. Try this paper without the following materials, but use them if they help**:** a centimeter ruler, a pair of scissors and a piece of thin cardboard such as a cereal box or file folder.

Here are five cardboard strips, each 1 widget wide (We don't really care what a widget is), and with the heights**:** 16 wd., 8 wd., 4 wd., 2 wd., and 1 wd. The smallest strip, of course, is a 1 wd. by 1 wd. square.


16 8 4 2 1

1) If you stack vertically the three shortest strips on top of the strip marked 8, what is the total height of the new stack? _______
2) What strips would you stack to make a 13 wd. height?__, __, __.
3) Which strips to make 7 ? ____, ____, ____
4) If you made one stack of all those *shown*, what would be the length of a single strip 1 wd higher than that stack? ______

(Your new strip's length should follow the pattern of the others.)

Correct exercises 1-4

5) Now you have a pattern of strips, each twice as long as the next, left to right. Gaze at them and mentally arrange strips to make stacks of each of the numbers 1 to 15. To make 6 for example, use a 4 strip and a 2 strip. For 10, use 8 and 2. Can each and every number up to 15 be made? Y/N _____
6) What is the greatest possible stack length if the 16 strip is used with the smaller ones in the combining? ____
7) In making every whole number from 1 to 31 with the numbers 16, 8, 4, 2, 1:
   a) Do you ever need the same strip twice to make any particular number? ____
   b) Do you always have the strips you need (no strip is missing) to make to make any particular number up to 31 ? ____
   c) Was there any strip you did *not* use at least once? ____
8) What is the length of all combined strips a) less than 8 ? ___ b) less than 16? ___
   c) less than 64 ____ d) including 64 ____ e) What is the next new strip after 64? ____

POWERS OF 2 AND THE MAGIC NUMBER CHART *Unit 4

Answers 1 – 8

1) 15 **2)** 8, 4, 1 **3)** 4, 2, 1 **4)** 32 **5)** Yes **6)** 31 **7a)** No **b)** Yes **c)** No

8a) 7 **b)** 15 **c)** 63 **d)** 127 **e)** 128

9) Recall that $2^3 = 8$.

a) $2^{\square} =$ 64 b) $2^{\square} =$ 128 c) $2^7 - 1 =$ ____ d) $2^8 =$_____ e) $2^9 =$ _____ f) $2^{\square}$ = 2048

Note: Exercise 9c should have the same answer as exercise 8d.

10a) Does $2^8 = 2^7 \times 2$? (Yes/No) _______ b) $2^8 =$________ c) $2^{10} =$ ___________

*11) If you had strips all the way up to and including 2^{10} , what number would they add up to?____________(You don't need to add to do this one.)

Notice how the numbers begin to get large quickly as the exponent of 2 gets larger.

12) Use your scientific calculator to compute 2^{15}. Press: 2, y^x, 15, = , _____________

(Your calculator might use other letters for the raising-to-power symbol, y^x.)

13) Use your calculator to find out whether $2^{10} \times 2^{13} = 2^{23}$.

Press 2, y^x, 10, x, 2, y^x, 13, = . Then see if 2^{23} equals that answer. Does it?_____

*14) If you had strips up to and including 2^{15}, what would they all add up to, end to end?

Hint: Use your calculator, but you don't need to add. ______________

We can get some rather surprising results by using a systematic way to record which strips we used in making any particular number.

Strips →	2^7= 128	2^6= 64	2^5= 32	2^4= 16	2^3= 8	2^2= 4	2^1= 2	1
Used or not → (1 = used, 0 = not)	0	0	1	0	1	1	0	0

The number 44 is represented: (32 + 8 + 4).

POWERS OF 2 AND THE MAGIC NUMBER CHART *Unit 4

15) As on previous page, write the number represented by these binary digits (Binary means "pertaining to two").

The number ____ is represented:

(___ + ___ + ___)

16) Write binary digits so the result will be 41. →

17) Now do 42 →

and 43 →

18a) Are you using Bing? Try it doing 44. →

$2^7=$ 128	$2^6=$ 64	$2^5=$ 32	$2^4=$ 16	$2^3=$ 8	$2^2=$ 4	$2^1=$ 2	1
1	0	0	1	0	0	1	1

Here is a simpler form for writing binary numbers: 11001_2 the little $_2$ is a subscript (sub for below) and says that this is a binary, or base two numeral. Of course, in this notation, you are not told what each place value is worth. <u>You</u> must provide that information.

18b) Express the value of 11001_2 in base ten. __________

19) Change these binary numerals to standard base ten numerals:

a) 1001_2 = ___ b) 11011_2 = ____ c) 100000_2 = ____ d) 111111_2 = ______

20) Change these base ten numerals to binary form. Don't forget the $_2$.

a) 13 = ______ b) 26 = ________ c) 126 = ___________ *d) 257 = _______________

Discount leading zeros

*21) How many times does a one appear in the binary numeral for 4,097 (This is close to a power of 2)? _____

*22) How many zeros appear in the binary numeral for 4095? _____

POWERS OF 2 AND THE MAGIC NUMBER CHART *Unit 4

Answers 9 – 22

9) a.^{6} **b.**^{7} **c.** 127 **d.** 256 **e.** 512 **f.** ^{11} (eleventh power) **10a)** Yes **b)** 256 **c)**1024

***11)** 2047 You could think: One less than 2^{11}.

12) 32,768 **13)** Yes. (8,388,608 for each)

14) 65,535 (1 less than 2^{16})

Memory aid: Note the coincidence that 10 begins the answer to 2^{10}.

		$2^7=$ 128	$2^6=$ 64	$2^5=$ 32	$2^4=$ 16	$2^3=$ 8	$2^2=$ 4	$2^1=$ 2	1
15) 128 + 16 + 2 + 1	147	1	0	0	1	0	0	1	1
	16) 41	0	0	1	0	1	0	0	1
	17) 42	0	0	1	0	1	0	1	0
	43	0	0	1	0	1	0	1	1
	18a) 44	0	0	1	0	1	1	0	0

18 b) 25

19) a. 9 **b.** 27 **c.** 32 **d.** 63

20) a. 1101_2 **b.** 11010_2 **c.** 1111110_2 **d.** 100000001_2

***21)** Twice, beginning and end. ***22)** None

23) **<u>Final problem:</u>** In exercise 15 you may have noticed that no entry was made in the very top row, last box of the table. According to the pattern in the top row, fill in the exponent: 2^{----} = 1 ?

We will *show* later that $2^0 = 1$.

The Magic Number Chart is on page 6 of this unit.

POWERS OF 2 AND THE MAGIC NUMBER CHART *Unit 4

A	B	C	D
1 = 1	2 = 10	4 = 100	8 = 1000
3 = 11	3 = 11	5 = 101	9 = 1001
5 = 101	6 = 110	6 = 110	10 = 1010
7 = 111	7 = 111	7 = 111	11 = 1010
9 = 1001	10 = 1010	12 = 1100	12 = 1100
11 = 1011	11 = 1011	13 = 1101	13 = 1101
13 = 1101	14 = 1110	14 = 1110	14 = 1110
15 = 1111	15 = 1111	15 = 1111	15 = 1111

Examining the chart above, notice that column A contains binary numbers with a one in the last, or ones place. Their base ten partners are the odd numbers 1 – 15.

Column B has binary numbers with a 1 in the twos place and their base ten partners.

Column C has binary numbers with a 1 in the fours place and their base ten partners.

Column D has binary numbers with a 1 in the eights place and their base ten partners.

Suppose you show this chart to someone and ask them to think of some number from 1 to 15. You then point to column A and ask "Is your number in column A?"

If the answer is "Yes" then you know that the number in binary form has a 1 in the ones place. Suppose the rest of the answers for columns B, C, and D are "No", "Yes", "No".

Now you know that the number has a 1 in the ones place (Column A) and a 1 in the fours place (column C). You immediately say "five" and you might be rewarded by a look of puzzlement or a "How did you do that?" question.

But that might not be easy to do. You will not have the binary numbers on the chart you display, so there is a chart on the next page which will help you to know what number to add mentally as you go along. This is done by having a 1 in the upper left corner of the first chart, a 2 in that same place on the second chart, a 4 in the third chart, etc.

Also, there are six charts on this page which allow you to have choices up to 63.

Be sure to practice using the page with a friend before going out into the world with your Magic Number Chart.

POWERS OF 2 AND THE MAGIC NUMBER CHART

*Unit 4

A

1	3	5	7
9	11	13	15
17	19	21	23
25	27	29	31
33	35	37	39
41	43	45	47
49	51	53	55
57	59	61	63

B

2	3	6	7
10	11	14	15
18	19	22	23
26	27	30	31
34	35	38	39
42	43	46	47
50	51	54	55
58	59	62	63

C

4	5	6	7
12	13	14	15
20	21	22	23
28	29	30	31
36	37	38	39
44	45	46	47
52	53	54	55
60	61	62	63

D

8	9	10	11
12	13	14	15
24	25	26	27
28	29	30	31
40	41	42	43
44	45	46	47
56	57	58	59
60	61	62	63

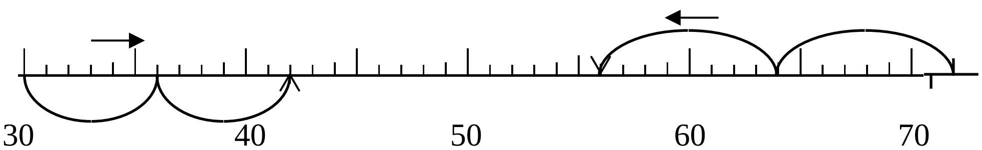
E

16	17	18	19
20	21	22	23
24	25	26	27
28	29	30	31
48	49	50	51
52	53	54	55
56	57	58	59
60	61	62	63

F

32	33	34	35
36	37	38	39
40	41	42	43
44	45	46	47
48	49	50	51
52	53	54	55
56	57	58	59
60	61	62	63

NUMBER LINE JUMPS 1 — *Unit 5



The +6 jumper moves to the right and the -8 (negative 8) jumper moves to the left. Either kind of jumper (+ or -) can be above or below the line. Here, the negative jumper is above the line.

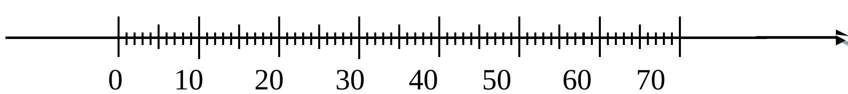
1a) Carefully draw five <u>more</u> jumps of the +6 jumper which started at 30.

b) Show five <u>more</u> jumps of the -8 jumper which started on 72.

c) What number(s) are common to both jumpers, that is, what numbers were hit by <u>both</u> jumpers? Consider a common starting or landing point as a hit. ______________

d) In part b), your -8 jumper should have landed (finally) on 16. Where would it have landed finally if it had started on 74 instead of 72?______

e) Your +6 jumper landed on 72 (or should have). Where would it have landed if it had started on 33 instead of 30? ______



No drawing necessary on this number line.

f) On the number line above, picture in your mind a +10 jumper starting at 0. It lands on 10, 20, 30, etc. Now picture in your mind the +10 jumper starting on 2. His next three landing places will be ___, ___, ___.

g) A +10 jumper starting on 36 will have what next 4 landing places? ___, ___, ___, ___.

The numbers 10, 20, 30, etc. are called 10 x n numbers because they are multiples of 10, that is, 10 x 1, 10 x 2, 10 x 3. We will call them simply 10n numbers and let the "x" sign be understood. This is commonly done in mathematics. Used here, n will be a whole number.

*h) What are the next four 10n numbers following 80? ______, ______, ______, ______

i) What are the next four 7n numbers following 70? _____,_____, _____, _____

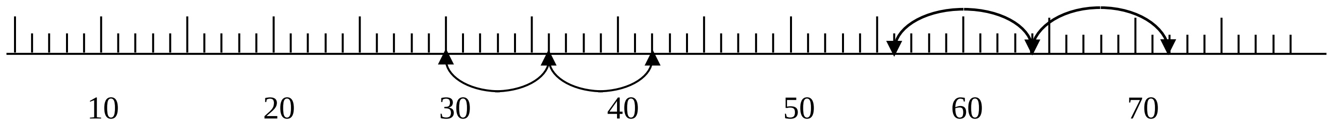
j) What is the next 5n number following 632? ____

*k) What are the next two 3n numbers following 100? ______, ______

*l) What are the next two 3n numbers following 6 x 10^5? _________, _________ (Use no exponent in answer.)

Correct answers to here. Answers are on page 5 of this unit.

NUMBER LINE JUMPS 1 | *Unit 5



The ***answers*** to f) on page 1 of this unit**:** 12, 22, 32, are not called 10n numbers even though they advance by ten, because they are not multiples of 10. They are called 10n + 2 numbers because each is two greater than a multiple of ten.

> The numbers in m) advance only 10 at a time. The numbers are landing places, not jumpers.

m) What are the next four 10n + 2 numbers following 32? ___, ___, ___, ___.

n) What are the next three 10n +6 numbers following 66? ___, ___, ___.

o) On the number line above, what is the largest 6n number represented by a tick mark ?____

p) What is the smallest two-digit 8n number represented on that same number line? _____

Check answers so far.

q) In part c), you should have found that 48 and 72 were hit by **both** jumpers. There is an even smaller number on the number line which is both a 6n and an 8n number, greater than 0. What is this **lowest** common multiple of 6 and 8? ____

Note: 7n is a landing place on the number line. It is a multiple of 7, such as 700 or 2,121.
A 7n + 2 number is 2 larger than a multiple of 7, such as 16 or 23 or 2,123.

2) What are the first four 7n + 2 numbers following 70? ____, ____, ____, ____.

Notice in exercise 2 that the four landing places are not multiples of 7, but they are landed upon by a +7 jumper. When landings are 7 apart, they are being hit by a + 7 jumper (or by a -7 jumper.)

3) An 8n + 3 number is 3 bigger than some 8n number. 35 is 8 x 4 + 3. 59 is 8 x 7 + 3.

a) 43 is 8 x ___ + ___ b) 75 is 8 x ___ + ___

c) What is the smallest 8n + 3 number greater than 80? _____ d) Greater than 86? _____

e) When any 8n + 3 number is divided by 8, what is the remainder? ____

4a) A +10 jumper starts on 21 and makes 4 jumps. What is its fourth landing place? ____

b) If it keeps on going will it ever land on 181?_____ On 14,671? _____ On 10^{90} ? _____

c) Will it ever land,314 on 16? ____ On 100,001 ? ____ On 3,748? ____ *On 10^{50} + 1 ? _____

Number Line Jumps 1 *Unit 5

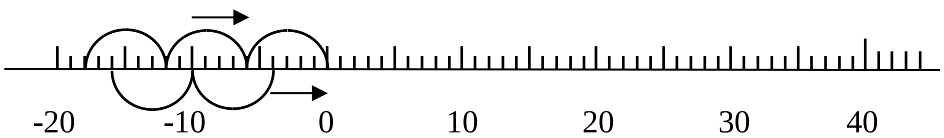
d) Will it ever land on a 10n number (a multiple of 10)? _____

5) A +8 jumper**:**

a) Starts on 0 and makes 3 jumps. Where does it end? _____

b) Starts on 2 and makes 3 jumps. Where does it end? _____

c) Starts on 12 and makes 5 jumps. List all of these 8n + 4 numbers hit. __________________



6a) Above the line, show 5 more jumps of the +6 jumper. Where does it land? ___

b) Below the line, show 6 more jumps of the +6 jumper. Where does it land? ___

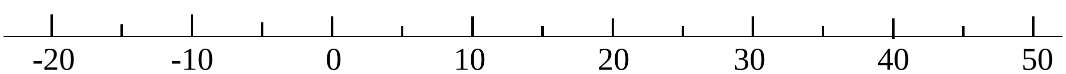
c) Above the line, each point hit is a 6n number. Below the line, each is a 6n +___number.

Note: Any 6n + 2 number is also a <u>6n – 4</u> number..

Example: <u>6 x 4</u> + 2 = 26 and <u>6 x 5</u> - 4 = 26

d) Likewise, any 9n + 6 can be regarded as a 9n __ __ number.

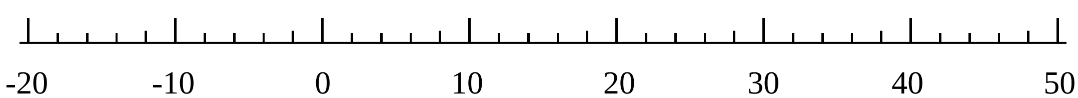
Correct Answers so far.



7a) A +5 jumper starts on –10 (negative 10) and makes 3 jumps. Where does it land? _____

b) <u>Show</u> above the line a -10 jumper making 5 jumps from 30. Lands where? _____

c) <u>Show</u> below the line a -10 jumper making 4 jumps from 33. Estimate locations. Lands at ___.



8) On the <u>above</u> number line, only the even numbers have tick marks.

a) A 4 jumper makes 2 jumps from 2. Lands where? _____

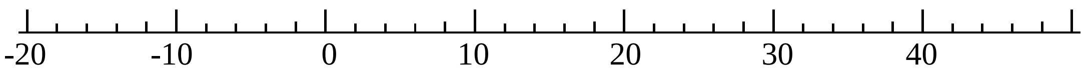
b) A -4 jumper makes 2 jumps from 2. Lands where? ______

c) <u>Show</u> above**:** 10 (start) + 3 jumps of 4 = _____ . (Landing point)

d) Show 50 (start) - (minus) 4 jumps of 10 = ______.

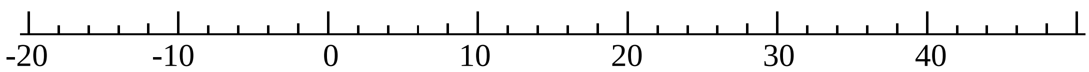
e) <u>Show</u> (below the above line): 0 - 2 jumps of 5 = _____ .

NUMBER LINE JUMPS 1 *Unit 5



9a) On the number line above, show atop the line, 3 jumps of -4 from -6 = _____

b) Show 10 + (3 x 4). This means start on 10 and make 3 jumps of 4 = _____



*10a) On the number line above, show 20 + (2 x 8). This is 2 jumps of 8 starting on 20 and lands on _____.

b) Show 10 + (2 x -8). This is 2 jumps of negative 8. Lands on _____.

c) Below the line, show -2 + (3 x -4). Lands on _____.

d) (Don't show) 100 + (50 x –2) = ___

11a) True or false: If a 6 jumper starts on a multiple of six, it will always hit 6n numbers. ___

b) T/F: If an 8 jumper does not start on an 8n number, it will never hit a multiple of 8. ___

*c) -10 + (3 x -5) = ____

d) A -2 jumper and also a -3 jumper both start on 10. Name the next 3 common points hit. (The next 3 *both hit)*

________ ________ ________

Recollections (Reminders: $4^3 = 4 \times 4 \times 4 = 64$; $7 \times 10^4 = 7 \times 10{,}000 = 70{,}000$)

12a) 2^5 = _____ b) 3^3 = _____ c) 4×10^5 = ________________ d) 2.389×10^2 = ____________

e) $489000 \div 10^{---} = 48.9$ f) $\left(\frac{2}{5}\right)^3$ = _______ *g) $\left(\frac{4}{5}\right)^{10000} \times \left(\frac{5}{4}\right)^{9999}$ = _____

13a) Write the next binary numeral following 10011_2 _____________

b) Change 10101_2 to base ten. ______ c) Change 35 to a binary base (2) numeral_______.

14) A +9 jumper starts on -2. Circle each of these numbers hit, assuming it goes beyond 909.

a) 18 b) 16 c) 20 d) 900 e) 898 f) 907 g) 909

NUMBER LINE JUMPS 1 — *Unit 5

Answers

1a) Landing places will be 48, 54, 60, 66, and 72 **b)** Landing places will be 48, 40, 32, 24, and 16 **c)** 48 and 72 **d)** 18 **e)** 75 **f)** 12, 22 32 **g)** 46, 56, 66, 76 **h)** 90, 100, 110, 120 **i)** 77, 84, 91, 98 **j)** 635 **k)** 102; 105 **l)** 60003; 60006 **m)** 42, 52, 62, 72 **n)** 76, 86, 96 **o)** 78 **p)** 16 **q)** 24

2) 72, 79, 86, 93

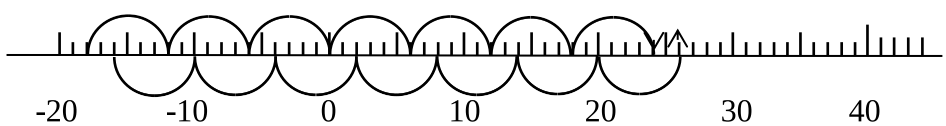
Keep this smaller than 8

3a) 5, 3 Of course, 43 could be thought of as 8 x 4 + 11 or even 8 x 3 + 19. But we keep the second number as large as possible so the third number will then be the remainder. **b**) 9 + 3 **c)** 83 **d)** 91 **e)** 3

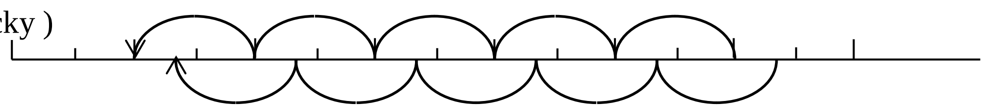
4a) 61 **b**) yes, yes, no **c)** no, yes, no, yes **d)** no

5a) 24 **b)** 26 **c)** 20, 28, 36, 44, 52

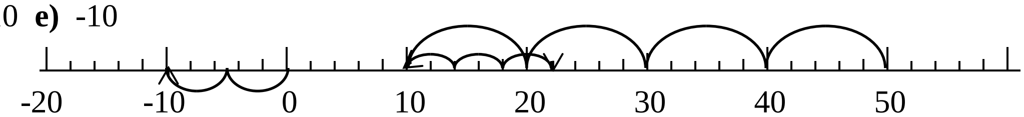
6a) 30 **b)** 32 **c)** 2 **d)** -3



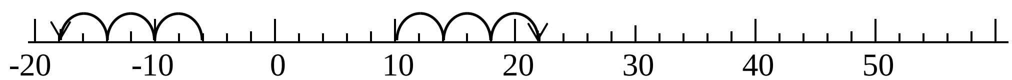
7a) 5 **b)** -20 **c)** -7 (Tricky)



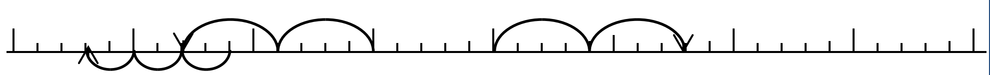
8a) 10 **b)** -6 **c)** 22 **d)**10 **e)** -10



9a) -18 **b)** 22



10a) 36 **b)** -6 **c)** -14 **d)** 0



11a) True **b)** True ***c)** -25 **d)** 4, -2, -8

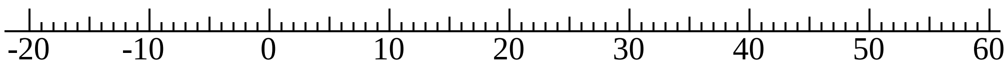
Recollection Answers

12a) 32 **b)** 27 **c)** 400,000 **d)** 238. 9 **e)** ^{4} **f)** 8/125 **g)** 4/5

13a) 10100_2 **b)** 21 **c)** 100011_2

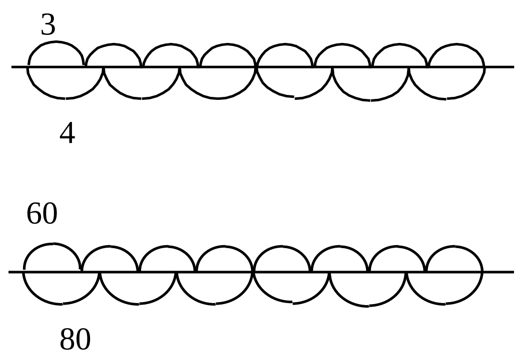
***14a)** Circle: **b)** 16 **e)** 898 **f)** 907

NUMBER LINE JUMPS 2 **Unit 6



The above number line is for reference only. No drawing is required

The jumper line to the right works for showing common multiples of 3 and 4. But since no scale of numbers is given on the line, we can use the diagram to show common multiples of any pairs of numbers. 30 and 40 work (3 tens to 4 tens), and so do 6 and 8. Any pair of numbers in the ratio 3 to 4 can use this diagram. "In the ratio 3 to 4" means that the numbers can be made into a fraction that reduces to 3/4.



Most can be done mentally

1) Use your ingenuity and circle each of the following pairs that equal 3 to 4, or 3/4; that is, chose those that are in the ratio 3 to 4 and therefore could use either jumper line above.

a) 6 and 8 b) 24 and 30 c) 24 and 32 d) 1200 and 1600 e) 1 1/2 and 2 f) 0.6 and 0.8

g) 2.1 and 2.8 h) $8\frac{1}{2}$ and 11 i) 2 and $2\frac{2}{3}$ j) (Be careful) 10^3 and 10^4 k) 6×10^5 and 8×10^5

*l) 12×10^{53} and 16×10^{53} m) .06 and .08

*2a) A +6 jumper and a +7 jumper, both starting on 20, will have their next two *common* landing points at _____ and ______. (Peek at this answer when done.)

b) A -2 jumper and a -3 jumper both start on 10. Name the next three common points hit. (Looking at the number line at the top might help.) _________________________

3a) A -10 jumper making 6 jumps starting at 40, ends on ____. b) 40 + (6 x -10) = _____

4) Starting on 0, what will be the first common landing for a), b) & c)? (Hint: Jump by the largest number first and each time check the others to see if they both land with the largest number.)

*a) 3, 4, and 6 _____ b) 2, 3, and 4 _____ c) 3, 5, and 10 _____

NUMBER LINE JUMPS 2 **Unit 6

Notice that exponents (in exercises 1 j, k & f) work in unexpected ways for common multiples. 1000 is a multiple of 100, so 10^3 is a multiple of 10^2, even though the exponent 3 is not a multiple of the exponent 2. Also, 10^{23} is a multiple of 10^{20} because 10^{23} has three more zeroes and is therefore 1000 *times* as big. ($10^{23} = 10^{20} \times \underline{10}^3$!!) So, 10^{23} is a multiple of 10^{20}.

Exercise 4, continued (looking for first common landing point of these numbers. Start is 0.):

e) 2000, 3000 and 4000 __________ f) 2,000, 50,000 and 100,000 _____________

g) 2000, 300 and 4000 ________ *h) 10^2, 10^3 and 10^4 ____ *i) 10^{15}, 10^{19}, 10^{20} and 10^{23} ____

Check answers before going on. There might be some surprises.

Don't forget the strategy of jumping the largest number first and then checking the others, unless, of course, you can see the answer easily.

5) What is the lowest common multiple (LCM) of : a) 4 and 9 _____ b) 4, 6, and 9 _____

c) 3, 6, and 9______ d) 40, 60, 90 ______ e) 8 and 18 ________ f) 2, 5, 7 ______

g) 1000, 2000, 6000 ________ h) 3 x 1000, 4 x 1000 and 6 x 1000 ______ *i) 3×10^3 & 5×10^4 6)______

6) A -4 and a -6 jumper both start on 30. Name all their common landing points before passing -35. ____

*7) Consider the numbers 2×10^3, 3×10^4 and 7×10^5. Any common multiple of these three numbers **must** be (T/F):

a) A multiple of 7 ____ b) A multiple of 10^4 ____ *c) A multiple of 10^{12} ___

d) A multiple of both 2 and 7 ____ e) A multiple of 10^{60} ____(Correct thoughtfully.)

**8) What is the LCM (lowest common multiple) of the three numbers in ex. 7? ______________

*9) What is the LCM of 1, 2, 3, 4, 6, 8, 12 and 24? ______________

*10) Remember to try *mentally*: $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{12}+\frac{1}{24}=$

NUMBER LINE JUMPS 2 **Unit 6

Answers

1) All should be circled except b), h), j)

2a) 62 and 104 **b)** 4, -2, -8

3a) –20 **b)** –20

4a) 12 **b)** 12 **c)** 30

e) 12,000 **f)** 100,000 **g)** 12000 **h)** 10^4 or 10000

***i)** 10^{23} or 100,000,000,000,000,000,000,000. Either is okay.

5a) 36 **b)** 36 **c)** 18 **d)** 360. Note its relation to part b.

e) 72 **f)** 70 **g)** 6000 **h)** 12000

i) 15 x 10^4 or 150000

6) 18, 6, -6, -18, -30

7a) T **b**) T

c) F; it <u>can</u> be a multiple of 10^{12} but does not <u>have</u> to be. **d)** T

e) F; but it <u>can</u> be.

Note: 10^5 can be divided evenly by any of 10^3, 10^4, or 10^5.

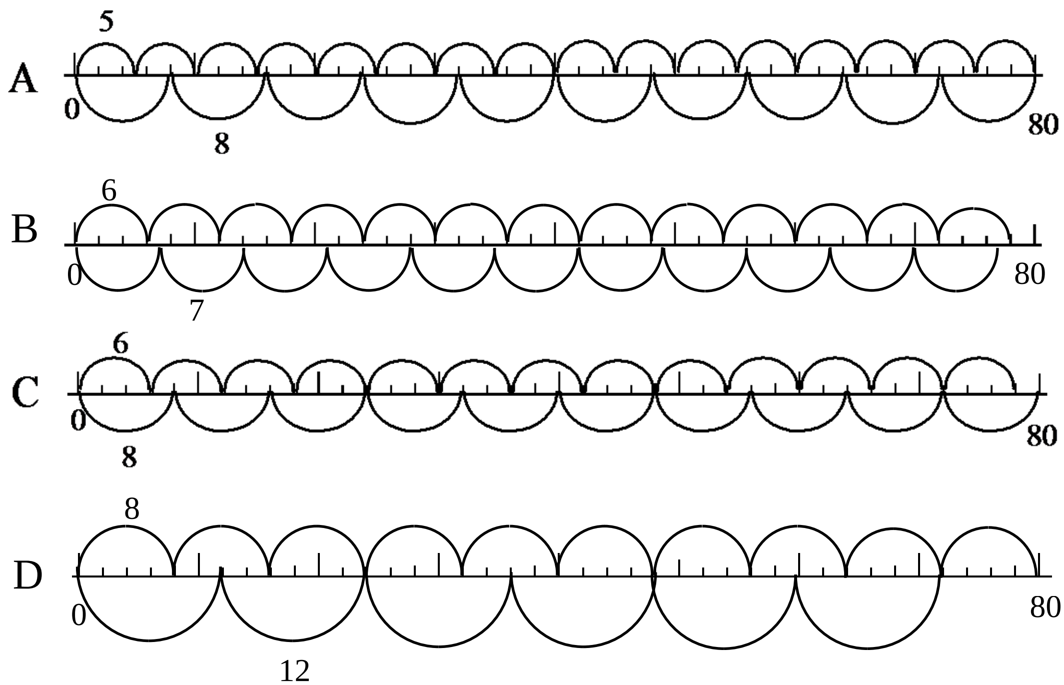
Note 2: It must be a multiple of 10^4, so it is T for True, but it must also be a multiple 10^5.

****8)** 42 x 10^5 or 4,200,000

9) 24

***10**) 36/24 = 3/2 or 1 1/2

NUMBER LINE JUMPS 3 **Unit 7



1a) Which number line above shows only one common multiple (excluding 0)? ____

b) What is that common multiple? ____

c) What is the next common multiple, beyond the end of the shown number line? ____

d) And the next two after that are _____ and _____

2a) Which line has exactly two common multiples showing? Do not count 0. ___

b) What are those two common multiples? ____ and ____

3) Note that in both number lines A and B, the <u>lowest</u> common multiple (LCM) is the <u>product</u> of the two jumper numbers.
Is this true in **C:** a) the pair 6, 8? ___ and in **D:** b) 8, 12? ___

You have heard of <u>factors</u>**:** factor x factor = product; or, factor x factor x factor = product, etc. You have heard of <u>primes</u>, too. In a prime number, if the only factors are 1 and the number itself, then the product is prime, as in 7 and 19. There are no other factors of 7 or 19 aside from 1 and itself.

NUMBER LINE JUMPS 3 **Unit 7

When the LCM of two numbers is their *product*, the numbers are *relatively prime.*

The term *relatively prime* is a good one. It describes two numbers which may or may not themselves be prime, but they share no factors other than 1, like 5 and 8, or 6 and 7, or 4 and 15. In each pair the numbers are "prime to each other". Numbers having **common factors** (sharing factors) are: 6 and 8 share 2, or 8 and 12 share 4 (and also share 2).

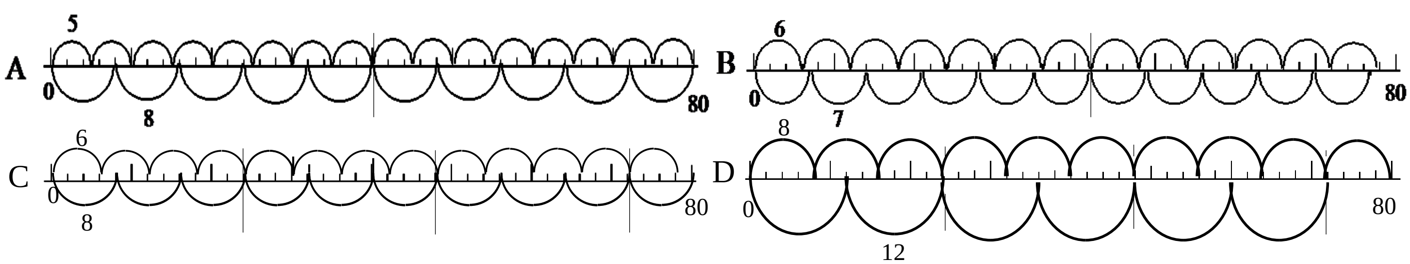
3) *(continued)* Circle the letter of each pair containing numbers which are relatively prime:

c) 5, 8 d) 9, 8 e) 6, 18 f) 6, 19 g) 4, 9 h) 20, 35 i) 105, 2035 j) 112, 213

k) 1313, 1315 l) 182, 1314 m) Any pair of consecutive even numbers.

n) Any pair of consecutive odd numbers.

4a) The LCM in **D** is ____. b) The other common multiples seen in **D** are ______ and ______.



5a) Which of the four number pairs in lines A - D are relatively prime? _____________

b) Name the LCM (lowest common multiple) of 6 and 8. ____

c) Name their higher two common multiples shown on number line C. ______, ______

d) The LCM on line D is _____. e) The other common multiples on D are _____, _____

6a) Name *all* the common multiples of 6 and 8 from their LCM to their product, inclusive, (including both the LCM and the product). _________________________________

b) Do the same for 8 and 12. ____

7a) (Y/N) Is 240 a common multiple of 6 and 8? _____; of 8 and 12? _____

b) (Y/N) Is 480 a common multiple of 6 and 8? _____; of 8 and 12? _____

*c) (T/F) Any multiple of a common multiple is a common multiple. ____

d) (T/F) For all pairs of whole numbers there is a GCM (greatest common multiple). ____

*e) What is the next common multiple of 8 and 12 after 480? _____

f) What is the largest common multiple of 8 and 12 below 480? _____

NUMBER LINE JUMPS 3 **Unit 7

*g) Name all common multiples of 8 and 12 between 900 and 1000. You could use b) and c) to help think of a shortcut strategy. ____

h) T/F In any interval of 28 on the number line (including endpoints), there is a common multiple of 4 and 7. ____

**i) What is the next common multiple of 4 and 7 after 28 x 10^4? ____________ (Do not use an exponent in your answer.)

Correct to here.

Exercise 7j - o are T/F and concern only the *positive* whole numbers.

j) Every pair of whole numbers greater than zero has an LCM. ____

k) The product of two numbers is always a common multiple of the two numbers. ____

l) The product of two numbers is always their LCM. ____

m) The product of two numbers is sometimes their LCM. ____

n) Two very large numbers necessarily share factors besides 1. ____

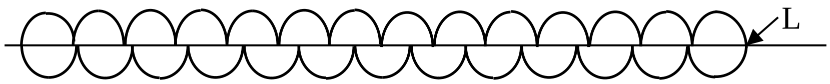
o) If any two positive, unequal jumpers both start at zero, they will have many common multiples. ____

p) Y/N. Does it seem to you that any two consecutive counting numbers are relatively prime? ____ This principle about consecutive numbers is important. More later.

Answers to 1 – 7

1a) B	**b)** 42	**c)** 84	**d)** 126 and 168	
2a) A	**b)** 40 and 80	**3a)** No **b)** No	**c-n)** circle the pairs:	c, d, f, g, j, k, n
4a) 24	**b)** 48, 72			
5a) 5 and 8, and 6 and 7		**b)** 24	**c)** 48 and 72	**d)** 24
e) 48, 72		**6a)** 24, 48	**b)** 24, 48, 72, 96	
7a) Yes, yes	**b)** Yes, yes	**c)** T	**d)** T --The whole numbers are unending.	
e) 504	**f)** 456	**g)** 912, 936, 960, 984		**h)** T
i) 280,028	**j)** T	**k)** T	**l)** F	**m)** T
n) F	**o)** T	**p)** Y (Think about 2 and 3)		

NUMBER LINE JUMPS 3 | **Unit 7

E 

In this pattern of jumps, we have made the smaller jumper (above the line) **one less** than the longer jumper below the line. You can see from diagram **E** how the smaller jumper gradually and steadily loses ground to the greater jumper, <u>one unit at a time</u>, until they finally meet at point L.

8) Can *any* consecutive pair of numbers share a factor aside from 1? ____

9)

	Numbers		**GCF**	**LCM**		
	A	**B**	**C**	**D**	**A x B**	**C x D**
a)	20	15	___	___	_______	_______
b)	8	6	___	___	_______	_______
c)	7	5	___	___	_______	_______
d)	12	16	___	___	_______	_______

See a relationship? Use it.

	A	B	C	D	A x B	C x D
e)	60	64	___	___	_______	_______
f)	___	40	___	___	2000	_______
g)	___	24	4	120	_______	_______
h)	30	__	___	___	960	_______

10) If the product of two numbers is 384 and their LCM is 48, their GFC is _____.

*11) Do these mentally if you can.

a) 100 and 105: GCF_________ and LCM ________

b) 300 and 500: GCF ___________ and LCM________

c) 3000 and 3003: GCF ________________ and LCM ________

12) May the difference between two numbers be their GCF? ______; *must* it? ______

13) Goldbach's conjecture: "Every even number greater than 2 can be expressed as the sum of 2 primes". Exercise : 16 = 13 + 3, or 11 + 5 Show that this is true for the following, but, for each number given, pick the two primes farthest apart, as 16 = 13 + 3. Do not pick 11 + 5.

60 = ___ + ___ 40 = ___ + ___ 98 = ___ + ___ *103 = ___ + ___

NUMBER LINE JUMPS 3 **Unit 7

14) Let $A_{20,17}$ mean 20 x 19 x 18 x 17 and likewise, $A_{14,12}$ means 14 x 13 x 12.

Compute: $$\frac{A_{15,9}}{A_{16,10}} \times 1\frac{1}{3} =$$

15) By how much does the LCM of 150 and 200 exceed their GCF? _____

*16) Circle each which belongs in the sequence 23, 32, 41, 50 . . . etc.

80,123 (9 x 7) + 25 1,002 and 2,001

17) Best if Done Mentally

	Kind of Jumper	Start	Number of Jumps	Lands
a)	3	1	400	
b)	5	100	3	
c)	5	99	4	
d)	-5	0	1	
e)	-5	-30	3	
f)	2 ½	4	2	
g)	2 ½	5		15
h)	6	-6		6
i)	6	-36	6	
j)	-10	10	3	
k)	5	-20	8	
l)	-4	-20	1	
m)	7	-15		13
n)	-5	15		-5
o)	10	-10		20
p)	8	0		56
q)	8	-8		56
r)		3	6	45
s)	80		20	1680
t)	200	-1000		1400
u)		-50	50	150
v)		500	50	0

	Kind of Jumper	Start	Number of Jumps	Lands
aa)	8	0		72
bb)	8	1		73
cc)	9	-81	9	
dd)	8	-64	8	
ee)	8	-64		64
ff)	7	-7		49
gg)	7		9	126
hh)	-8	-10	3	
ii)	-8	80		-8
jj)	-9	100		-800
kk)	-8	71	9	
ll)	-8		9	-2
mm)	-8		9	0
nn)	-8		9	1
oo)	3		100	400
pp)	3	99		699
qq)	3	99		996
rr)	99	99	100	
ss)	1001	156	3	
tt)	4004		10	40041
uu)	25		6	300
vv)	24	150	6	

NUMBER LINE JUMPS 3 **Unit 7

18) Circle each of the following that belong in the sequence 47, 55, 63 . . .,

a) 80 b) 799 c) 30 x 8 - 1 d) 8000 + 8 e) 8000 – 1 f) 8 x 165 + 47 g) 807

19) A +8 and a +6 jumper each start at 70. What next three numbers will they both hit?

______, ______, _______.

20) An a +7 jumper starts from an unknown location less than 200. In the interval from 450 to 465 inclusive, the +7 jumper will land no less than ____ times and no more than ____ times.

*21) A +3 jumper starts on 201 and a -10 jumper starts on 352. Name every number that they *both* hit. There are five of them. ______ ______ ______ ______ ______

> Reminder: If the sum of the digits of a number is divisible by 3, so is the number.

Answers 8 – 21

8) Of course not.

(Below, for exercise 9, note that answers are not underlined.)

9)

	Numbers A	B	GCF C	LCM D	A x B	C x D
a)	20	15	5	60	300	300
b)	8	6	2	24	48	48
c)	7	5	1	35	35	35
d)	12	16	4	48	192	192

See a relationship? Use it.

	Numbers A	B	GCF C	LCM D	A x B	C x D
e)	60	64	4	960	3840	3840
f)	50	40	10	200	2000	2000
g)	20	24	4	120	480	480
h)	30	32	2	480	960	960

10) 8

11a) 5, 2100 **b)** 100, 1500 **c)** 3, 300,300 **12)** yes, no

13) 7 + 53, 37 + 3, 19 + 79, 101 + 2 **14)** ¾ **15)** 550 **16)** 80,123 (only 9n + 5 nos.)

NUMBER LINE JUMPS 3 **Unit 7

17)

	Kind of Jumper	Start	Number of Jumps	Lands
a)	3	1	400	***1201***
b)	5	100	3	***115***
c)	5	99	4	***119***
d)	-5	0	1	***-5***
e)	-5	-30	3	***-45***
f)	2 ½	4	2	***9***
g)	2 ½	5	***4***	15
h)	6	-6	***2***	6
i)	6	-36	6	***0***
j)	-10	10	3	***-20***
k)	5	-20	8	***20***
l)	-4	-20	1	***-24***
m)	7	-15	***4***	13
n)	-5	15	***4***	-5
o)	10	-10	***3***	20
p)	8	0	***7***	56
q)	8	-8	***8***	56
r)	***7***	3	6	45
s)	80	***80***	20	1680
t)	200	-1000	***12***	1400
u)	***4***	-50	50	150
v)	***-10***	500	50	0

	Kind of Jumper	Start	Number of Jumps	Lands
aa)	8	0	***9***	72
bb)	8	1	***9***	73
cc)	9	-81	9	***0***
dd)	8	-64	8	***0***
ee)	8	-64	***16***	64
ff)	7	-7	***8***	49
gg)	7	***63***	9	126
hh)	-8	-10	3	***-34***
ii)	-8	80	***11***	-8
jj)	-9	100	***100***	-800
kk)	-8	71	9	***-1***
ll)	-8	***70***	9	-2
mm)	-8	***72***	9	0
nn)	-8	***73***	9	1
oo)	3	***100***	100	400
pp)	3	99	***200***	699
qq)	3	99	**299**	996
rr)	99	99	100	***9999***
ss)	1001	156	3	***3159***
tt)	4004	***1***	10	40041
uu)	25	***150***	6	300
vv)	24	150	6	***294***

18) b, c, e, f, g

19) 94, 118, 142

20) 2, 3

21) 342, 312, 282, 252, 222

THOSE NINES Unit 8

Lots of patterns:

1

2 x 9 = 18
3 x 9 = 27
4 x 9 = 36
5 x 9 = 45
6 x 9 = ___
7 x 9 = ___
8 x 9 = ___
9 x 9 = ___
10 x 9 = ___
11 x 9 = ___

See what's happening in the answers as you go down each column. ↓ ↓ ↓ Now complete the pattern.

2} Here are some nine fact answers.

81 54 18

Add the digits in any answer. The sum is ___ .

Now write six other nine fact answers less than 99 (just the answers). Don't try to put them in any special order.

____ ____ ____ ____ ____ ____

3

Another pattern

4 x 9 = 36 8 x 9 = 72 5 x 9 = 45
↑ ↑ ↑ ↑ ↑ ↑

See the pattern? Do these:

6 x 9 = __4; 3 x 9 = __7; 7 x 9 = __3; __x 9 =36

__x 9 = 72 __x 9 = 81 __x 9 = 63 __x __= 54

Circle all the numbers which are answers to nines facts:

56 89 54 73 81 36 42

45 27 63 24 32 18

48 108 72 49 90

{5} Reminder:

8 x 9 = __2; 6 x 9 = __4
↑ ↑ ↑ ↑

[6] Use patterns to do these:

9 x 4 = ___ 8 x 9 = ___ 4 x 9 = ___

6 x 9 = ___ 9 x 3 = ___ 9 x 7 = ___

9 x 9 = ___ 2 x 9 = ___ 5 x 9 = ___

9 x 8 = ___ 7 x 9 = ___ 9 x 6 = ___

Nines, mixed with easy facts: ↓

(10)

Go across

1) 8 x 9 = ___ 2) 4 x 2 = ___ 3) 3 x 9 = ___ 4) 7 x 1 = ___ 5) 7 x 9 = ___

6) 7 x 3 = ___ 7) 9 x 7 = ___ 8) 2 x 3 = ___ 9) 10 x 9 = ___ 10) 9 x 9 = ___

12) 9 x 8 = ___ 13) 2 x 0 = ___ 14) 6 x 9 = ___ 15) 9 x 3 = ___ 16) 9 x 6 = ___

17) 4 x 4 = ___ 18) 9 x 8 = ___ 19) 5 x 9 = ___ 20) 9 x 4 = ___ 21) 9 x 2 = ___

22) 9 x 5 = ___ 23) 8 x 9 = ___ 24) 4 x 9 = ___

25)
  9000
x 600

Check all answers so far with special attention to exercise 25.
Note the number of zeros and do 26 and 27 mentally:

26) 70 x 900 = __________ 27) 400 x 90,000 =________________

THOSE NINES Unit 8

Exercises:

1) 90 x 80,000 = ______________ 2) 900 x 7,000 = ___________ = 63 x $10^{\square}$

3) 30 x 90 = ____ x $10^{\square}$ 4) 50 x 80 (Be careful) = 4 x $10^{\square}$

5) ____ x 80 = 7,200 6) _______ x 900 = 72 x 10^4

7) 3000 x ____ = 27 x 10^4 8) 3 x 9 x 10^2 = ________

*9)______ x 50 = 45 x 10^6 10) 700 x ______ = 63 x 10^4

11) 3000 x 9 x 10^2 = 27 x $10^{\square}$ 12) 90,000 x 9 x 10^2 = ____ x $10^{\square}$ Check answers.

*13) 300 x 9 x $10^{\square}$ = ___ x 10^4 14) 50 x _______ (Careful) = 30,000

*15) 50 x 8 x 10^4 = 4 x $10^{\square}$ *16) 400 x _____ = 2 x 10^5

After writing your answers for 13-16, mentally check them before looking at answers.

Recollections

Recall from the earlier paper Number Giants 2, that 27.801 x 10^2 = 2780.1 and that 27.801÷ 10^2 = .27801 Multiplying by a power of ten moves the point to the right (or the number to the left), and dividing by a power of ten makes the opposite move.

Supply any missing operation symbol or number or both (including exponent):

1) 3.093 x 10^4 = __________ 2) 3.039 ÷ 10^4 = ____________ 3) 675 $\square$ 10^2 = 6.75
Operation

4) .0234 x $10^{\square}$ = 234

*5a) _______________ x 10^4 = 234432. 5b) _____________ ÷ 10^4 = 234.432

6) Write the next three binary (base two) numerals following

110010_2 ____________, ____________, ____________

THOSE NINES

Unit 8

Answers for page 1 of this unit

2 x 9 = 18
3 x 9 = 27
4 x 9 = 36
5 x 9 = 45
6 x 9 = **5**4
7 x 9 = **6**3
8 x 9 = **7**2
9 x 9 = **8**1
10 x 9 = **9**0
11 x 9 = **9**9

See what's happening in the answers as you go down each column.
↓ Now complete
↓ the pattern.
↓

1

2} Here are some nines fact answers.
81 54 18
Add the digits in any answer. The sum is **9**.
Now write six other nines fact answers less than 99 (just the answers). Don't try to put them in any special order. **Any 6 of:**

90, 72, 63, 45, 36, 27, 9, 0.

3

Another pattern
4 x 9 = 36   8 x 9 = 72   5 x 9 = 45
↑ ↑ ↑ ↑ ↑ ↑
See the pattern? Do these:

6 x 9 = **5**4; 3 x 9 = **2**7; 7 x 9 = **6**3; **4**x 9 =36

8 x 9 = 72 **9** x 9 = 81 **7**x 9 = 63 **6** x **9**= 54

Circle all the numbers which are answers to nines facts:

56 89 54 73 81 36 42
45 27 63 24 32 18
48 108 72 49 90

{5} Reminder:
8 x 9 = **7**2; 6 x 9 = **5**4
↑ ↑ ↑ ↑

[6]
Use patterns to do these:
9 x 4 = **36** 8 x 9 = **72** 4 x 9 = **36**
6 x 9 = **54** 9 x 3 = **27** 9 x 7 = **63**
9 x 9 = **81** 2 x 9 = **18** 5 x 9 = **45**
9 x 8 = **72** 7 x 9 = **63** 9 x 6 = **54**

Nines, mixed with easy facts: ↓

(10) Go across

1) 8 x 9 = **72** **2)** 4 x 2 = **8** **3)** 3 x 9 = **27** **4)** 7 x 1 = **7** **5)** 7 x 9 = **63**
6) 7 x 3 = **21** **7)** 9 x 7 = **63** **8)** 2 x 3 = **6** **9)** 10 x 9 = **90** **10)** 9 x 9 = **81**
12) 9 x 8 = **72** **13)** 2 x 0 = **0** **14)** 6 x 9 = **54** **15)** 9 x 3 = **27** **16)** 9 x 6 = **54**
17) 4 x 4 = **16** **18)** 9 x 8 = **72** **19)** 5 x 9 = **45** **20)** 9 x 4 = **36**
21) 9 x 2 = **18** **22)** 9 x 5 = **45** **23)** 8 x 9 = **72** **24)** 4 x 9 = **36**

25)
 9000
x 600
5,400,000
or 54 x 10^5

Note the number of zeros after the 54 and do these mentally:

26) 70 x 900 = **63,000** or 63 x 10^3 **27)** 400 x 90,000 = **36,000,000** or 36 x 10^6

THOSE NINES | Unit 8

Answers for page 2 of this unit

Answers are acceptable either in exponent form or in standard form.

1) 7,200,000 or 72×10^5 **2)** 6,300,000 or 63×10^5 **3)** 27, ^{2} **4)** 4×10^3 **5)** 90 or 9×10^1 **6)** 800 or 8×10^2 **7)** 90 or 9×10^1 or 9×10 **8)** 2700 or 27×10^2

***9)** 900,000 9×10^5 **10)** 900 9×10^2 **11)** ^{5} **12)** 81 ^{6}

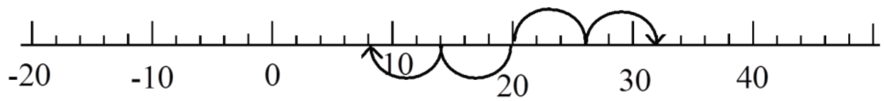
13) ^{2}, 27 **14)** 600 ***15)** ^{6} ***16)** 500 or 5×10^2

Recollection Answers

1) 30,930 **2)** .0003039 **3)** ÷ **4)** ^{4} **5a)** 23.4432 **b)** 2,344,320

6) 110011_2, 110100_2, 110101_2

("Quasi" means "in some sense" or "sort of")



1) One of the four parts of this problem is probably not fair. If you find it so, answer NF.

a) 20 + (2 x +6) = ___ b) 20 + (2 x -6) = ___ c) 20 - (2 x -6) = ___ d) 20 - (2 x +6) = ___

You probably answered NF to part c. (See figure below left.)

Look for a pattern in the four parts of exercise 1 as shown below.

a) 20 + (2 x +6) = 32
b) 20 + (2 x -6) = 8
c) 20 - (2 x -6) = NF
d) 20 - (2 x +6) = 8

At left, maybe the answer to **1c)** should be 32 to make two answers of 32 as well as two answers of 8. Maybe not. This "balancing" does not make a totally convincing argument that 20 - (2 x -6) = 32.

We could also note that when we **add negative** jumps as in b), the answer is <u>less</u> than the <u>starting</u> point. So, when we **subtract a negative** jumps as in c), perhaps we should get an answer <u>greater</u> than the starting point. Thus, 20 - (2 x -6) = 32. Perhaps, but still not definite.

In example 2 we will deal with *single* jumps and look at a third argument.

2a) How far is it from –20 to 40? (See number line above) _____

b) How far is it from 61 to 93? _____

In part b) you probably subtracted 61 from 93 to get 32, but in part a) you might have, or certainly *could* have <u>added</u> 20 and 40 to get 60. The number line suggests this. But subtraction is suggested by distance from one point to another. We think of it as a <u>difference</u>:

<u>93 - 61 = 32, and 40 - (–20) = 60</u>

40 - (-20) = 60 and 40 + (+20) = 60. Subtracting negative 20 is the same as adding positive 20.

A fourth, final and quasi-indisputable argument follows:

(-20) - (-20) = 0. Reason: any number minus itself = zero.

(-20) + (+20) = 0. Reason: easily demonstrated on the number line.

(-20) - (-20) = (-20) + (+20). Reason: each is equal to zero, so they equal each other.

We will take it as true that subtracting a negative number is the same as adding a positive number.

The answer to 1c) above (subtracting negative) is the same as *adding the positive* jumps in 1a) 32

Do all work mentally:

Subtracting negative jumps goes forward (higher).

3a) 50 - (2 x -5) = ____ b) 50 + (2 x +5) = ____ c) 50 - (2 x +5) = ____ d) 50 + (2 x -5) = ___

4) Only a single jump in each case.

a) 70 - (+8) = ___ b) 70 - (-8) = ___ c) 70 + (+8) = ___ d) 70 - (-10) = ___

Check all answers before going on. See page 3. Understand your errors, if any.

*5) Starting at -20 on the number line:

a) -20 - (-10) = ____ b) -20 - (+10) = ____ c) -20 + (+10) = ____ d) -20 + (-10) = ____

Check answers

6a) 64 + (-8) = ____ b) 64 - (-8) = ____ c) 64 + (+8) = ____ d) 64 - (-10) = ____

Check answers

7a) T/F: Subtracting a negative number = adding that positive number. _____

b) T/F: Subtracting a positive number = adding that negative number.____

c) 12 - (-3) = ____ d) 1000 - (-1000) = _______ *e) -1000 - (-1000) = _________

f) T/F: 1000 + (+1000) = 1000 + 1000 ___. (This means that you can usually omit the plus sign of a positive number. The answer, of course, is true.)

8a) 2000 + (-1) = ____ b) -2000 + (-1) = ____ c) 2000 - (-1) = ____ d) -2000 - (-1) = ____

9a) 1 - (+2000) = 1 - 2000 = ______ b) 0 - 2000 = ______ *c) -1 - 2000 = ______

d) -1 - (-2000) = ________ Think of number line jumps if you are unsure.

Check answers to exercises 7 - 9

10a) ____ + (-3) = 20 b) ____ + 3 = 20 c) -3 + ____ = -23 d) -3 - ____ = -23

e) ____ - 6 = 26 f) ____ - 6 = -26 g) $21\frac{1}{2}$ - _____ = 21 h) $21\frac{1}{2}$ - ____ = 22

i) 6 - 5 = ____ j) 5 - 6 = ____ k) 998 - 1000 = ____ l) 1000 - 998 = _____

Use no exponents in answers to exercises 11 - 17

11) 10^3 - (-3) = _____ 12) -3 - 10^3 = _____ 13) 4.7 - (-2.2) = _____ 14) 4.7 - (-2.3) = _____

15) (6×10^3) - (5×10^2) = ________ 16) (5×10^2) - (6×10^3) = _______

17) (6×10^2) - (5×10^3) = ________

There are two important ideas that you may have noticed, especially useful for exercises 16 and 17;

1. When you subtract a larger number from a smaller, the answer is negative, and to do this,
2. Subtract the smaller from the larger but then call the answer negative. So,

   3 - 5 = the opposite of 5 - 3, that is, 3 - 5 = -(5 - 3) = -2. Ponder this.

Answers

Probably not fair yet, but soon.

1a) 32 **b)** 8 **c)** 32 **d)** 8

2a) 60 **b)** 32

3a) 60 **b)** 60 **c)** 40 **d)** 40

4a) 62 **b)** 78 **c)** 78 **d)** 80

5a) -10 **b)** -30 **c)** -10 **d)** -30

6a) 56 **b)** 72 **c)** 72 **d)** 74

7a) T **b)** T **c)** 15 **d)** 2000 **e)** 0 **f)** T

8a) 1999 **b)** -2001 **c)** 2001 **d)** -1999

9a) -1999 **b)** -2000 **c)** -2001 **d)** 1999

10a) 23 **b)** 17 **c)** -20 **d)** 20 **e)** 32 **f)** -20 **g)** 1/2 **h)** -1/2 **i)** 1 **j)** -1 **k)** -2 **l)** 2

11) 1003

12) -1003

13) 6.9

14) 7

15) 5500

16) -5500

17) -4400

NEGATIVE EXPONENTS INVESTIGATED *Unit 10

1a) T/F: (3 x 2) x 4 = 3 x (2 x 4) _______ A "True" answer correctly says that it does not matter how we associate these numbers into pairs when we multiply them.

Likewise:

1b) 1.0125 x 10^2 x 10^3 = (1.0125 x 10^2) x 10^3 = ___________ x 10^3 = _____________.

or, 1.0125 x 10^2 x 10^3 = 1.0125 x (10^2 x 10^3) = 1.0125 x ____ = _____________.

Seeing this, we say that multiplication is ***associative.*** We can associate either the first two numbers or the second and third.

Also notice that 10^2 x 10^3 ≠ (does not equal) 10^6.The *exponents* are added. This multiplies the tens because:

100 x 1000 = 100,000

And 10^2 x 10^3 = 10^5

Confirm this mentally

Also, 100,000 ÷1000 = 100

$10^5 \div 10^3 = 10^2$ The exponents are subtracted. This divides the numbers.

1c) Besides multiplication, name the other operation in mathematics that is associative_____.

note

1d) .02 x 10^2 = _____ *1e) Three hundred ten-thousandths ÷ 10^2 = ________

no dash

*1f) Three hundred ten thousandths ÷ 10^2 = ___________

Seven hundred ten-thousandths = .0700 Seven hundred ten thousandths = .710

There are two "Not fair" exercises in 2 to 13. Answer them NF.

*2a) 3.045456 x 10^3 x $10^{\square}$ = 30,454,560 b) 30.45456 x $10^{\square}$ = 3,045,4560

*3a) 120.456 x $10^{\square}$ x 10^2 = 120456 b) 120.456 x $10^{\square}$ = 120456

4) 3.62 x 10^{200} ÷ 10^{197} = ___________ *5) .0321 x $10^{\square}$ ÷ 10^5 = .000321

6) 163.28 x 10^4 ÷ 10^3 = __________ 7) 163.28 x 10^4 ÷ 10^2 = __________

Check answers. All answers begin on page 9 of this Unit.

NEGATIVE EXPONENTS INVESTIGATED *Unit 10

In exercise 4 we saw competition between the powers of ten. The multiplication part beat the division part by 3, moving the number three places larger. 3.62 became 3620. This domination by an exponent is opposite in exercise 5: .0321 x $10^{\square}$ ÷ 10^5 = .000321, the <u>multiplication</u> exponent must be 2 smaller than the division exponent to make the answer 2 places smaller. The multiplication exponent of ten needs to be 3.

8a) 60.42 x $10^{\square}$ = .06042 b) 60.42 x $10^{\square}$ ÷ 10^7 = .06042

9) 60.42 ÷ $10^{\square}$ = .0006042 10a) __________ ÷ 10^5 = 3.2 b) ________ x 10^3 = 3.2

11a) 38.6 ÷ 10^{50} x 10^{51} = ___________ 11b) In a), does it matter how we associate?_____

12) 103.5 ÷ $10^{\square}$ = 103,500

*13) 16 million ÷ $10^{\square}$ = 16 thousand<u>ths</u>. 14) Six hundred ten-thousandths x 10^5 =________

15) This is <u>not</u> the same as exercise 14: Six hundred ten thousandths x 10^5 = ______________

The last time you met “Not fair” was the paper Quasi-Mathematical Arguments, where <u>subtracting</u> a <u>negative</u> number meant <u>adding</u> a <u>positive</u>, and <u>adding</u> a <u>negative</u> was the same as <u>subtracting</u> a <u>positive</u>.

In this paper we find exercise 8a, 60.42 x $10^{\square}$ = .06042, asking us to <u>multiply</u> by a power of ten and get an answer <u>smaller</u> than the number we started with. <u>Not fair</u>, but not impossible. Perhaps you are already guessing about negative exponents.

Also “Not Fair” exercise 12, asks us to <u>divide</u> 1.035 by a power of ten and get an answer *larger* than 1.035, in this case 103500. This might remind you of *subtracting a negative* from a number and getting an answer <u>larger than that number</u>, as in 10 - (-13) = 23.

NEGATIVE EXPONENTS INVESTIGATED

*Unit 10

Let's investigate. Follow the pattern:

16a) $23.35 \times 10^3 =$ _______ b) $23.35 \times 10^2 =$ ________ c) $23.35 \times 10^1 =$ _______

d) $23.35 \times 10^0 =$ _______ e) $23.35 \times 10^{-1} =$ ________ f) $23.35 \times 10^{-2} =$ ______

g) $23.35 \times 10^{-3} =$ _______ h) (Division) $23.35 \div 10^3 =$ ________

Check answers.

Exercise 16d) shows that **$10^0 = 1$**. The point does not move. Also, 16g) and h) have the same answer. This shows what you may already have suspected:

> **Multiplying by a negative power** of ten gives the same result as **dividing by a positive power** of ten.

17) More briefly, $10^4 \times 10^{-4} = 10^0 = 1$. Also, $10^4 \div 10^4 = 10^0 = 1$

a) $10^4 \times 10^3 = 10^{\square}$ b) $10^4 \times 10^2 = 10^{\square}$ c) $10^4 \times 10^1 = 10^{\square}$ d) $10^4 \times 10^0 = 10^{\square}$

e) If your answer to exercise17d was 10^4, then again, 10^0 must equal ___, the only number that can be multiplied by 10^4 to get 10^4 for an answer. Yes, **$10^0 = 1.$**

Continuing the pattern:

17f) $10^4 \times 10^{-1} = 10^{\square}$ g) $10^4 \times 10^{-2} = 10^{\square}$ h) $10^4 \times 10^{-3} = 10^{\square}$ i) $10^4 \times 10^{-4} = 10^{\square}$ =___

j) $10^4 \div 10^4 = 10^{\square}$ = ___ k) $8^4 \times 8^{-4} = 8^{\square}$ = __ l) $8^4 \div 8^4 =$ ______ m) $956^0 =$ _______

18a) $60.42 \div 10^{\square} = .06042$ b) $60.42 \times 10^{\square} = .06042$

Check all answers to here.

NEGATIVE EXPONENTS INVESTIGATED *Unit 10

19a) $6.5 \div 10^{\square} = 6500$ b) $6.5 \text{x} 10^{\square} = 6500$ c) $6.5 \div 10^{\square} = 6.5$

20a) $6.5 \div 10^{\square} = .0065$ b) $6.5 \text{ x } 10^{\square} = .0065$ c) $6.5 \text{ x } 10^{\square} = 6.5$

Check answers through exercise 20

21) Recall: a) 10 + (-18) = ____ b) 10 – (-18) = ____ c) 10 – (18) = ____

22) More recall: a) What is the likely *wrong* answer to $6 \div \frac{1}{2}$? ______

b) What is the correct answer to $6 \div \frac{1}{2}$? ______ c) (Perhaps) mentally, 18.42 ÷ .01 = ______

d) 18.42 x .01 = ________ e) $47.23 \div 10^{-3}$ = __________

Check answers through exercise 22

We have seen that it is perfectly possible to make larger by subtracting, as in 10 – (-18) = 28, and to make larger by dividing as in $6 \div \frac{1}{2} = 12$ and $47.23 \div \frac{1}{1000} = 47.23 \text{ x } \frac{1000}{1} = 47230$. This is the same as

$47.23 \div 10^{-3} =$
$47.23 \text{ x } 10^{3} = 47,230.$

23) Below, each pair is related by double opposite. Give answers.

. a) 16 + (+4) = ____ c) $33.47 \text{ x } 10^{2}$ = _______ e) $6 \div \frac{2}{3}$ = _____

b) 16 - (-4) = ____ d) $33.47 \div 10^{-2}$ = _______ f) $6 \text{ x } \frac{3}{2}$ = _____

In exercise 23, a) and b) give equal answers because of opposites, actually a double opposite: opposite numbers: **4 and –4**, and opposite operations: **addition and subtraction**. A double opposite also occurs in c) and d): opposite numbers (reciprocals): 10^2 and 10^{-2}, and opposite operations: multiplication and division. The same double opposite occurs in e) and f).

Check answers before going on.

NEGATIVE EXPONENTS INVESTIGATED *Unit 10

We already know that +4 and –4 are additive opposites. They lie opposite each other on the number line, and 4 + -4 = 0. A special fact about 0 is: 4 + 0 = 4.

A number retains its identity when added to 0.
0 is the **identity element** for addition.
Any number + 0 = that same number

Is there an **identity element** for multiplication such that the **identity times any number equals that same number**? "One" you say? Correct. 1 x 9 = 9 x 1 = 9 345.6 x 1 = 345.6
A mathematician would say:

For all numbers n,
n + 0 = 0 + n = n
and
n x 1 = 1 x n = n

For all numbers a and b,
If a + b = 0 then a and b are additive opposites.
If a x b = 1 then a and b are multiplicative opposites, or reciprocals.

A **Multiplicative** opposite, a kind of tongue twister, has a more popular name**:**

reciprocal (ree-sip-ro-cl)

The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, of 5 is $\frac{1}{5}$, of .3 is $\frac{10}{3}$ or $3\frac{1}{3}$.

Each of the pairs above **multiplies to get 1.**

REMINDER:
$10 \div \frac{2}{3} = 10 \times \frac{3}{2}$ because of double opposites: **opposite operation**, **opposite number.**
15 – (-9) = 15 + (+9) because of double opposites (opposite operation and number).

24) What is the reciprocal of 0? $\boxed{}$

Ask instead, what is the *multiplicative opposite* of zero? That is, what number, multiplied by zero, equals 1? $\boxed{} \times 0 = 1$ Answer: There is no such number, because any number times zero equals zero, not 1. Zero has no reciprocal.

NEGATIVE EXPONENTS INVESTIGATED *Unit 10

*** Be patient. Accept difficulty calmly. Return later if needed.

Agreements for these 2 pages only: O_p means additive opposite: O_p 6 = -6;
R means reciprocal of;
Read O_p O_p as opposite of opposite of.

Note: Having identity and opposites allows mathematics to be a powerful tool. They may seem trivial but they are not.

**25) True or False

a) The opposite of 5 = -5 ____

b) The multiplicative opposite of 5 = $\frac{1}{5}$ ____

c) - (-5) = 5 ___

d) - (-(-5)) =5 ___

e) -(-(-(-5))) = 5___

Check answers to e) now

f) -(-(-(-(-5))))= -5 __

g) -(-(-(-(-(-(-(-(-(-5))))))))))= 5 ___

h) R 5 = -5 ___

i) O_p 5 = -5 ___

j) R 5 = $\frac{1}{5}$ ___

k) O_p O_p -5 = -5___

l) O_p $-\frac{1}{3} = \frac{1}{3}$ ___

m) R $\frac{1}{3}$ = 3 ___

n) R 5 = $\frac{1}{5}$ ___

o) $4 = 4^1$ ____

p) $10^3 \times 10^{-2}$ = ___

q) $10^2 \times 10^{-2}$= ___

Check answers to q) now

r) R $4^2 = 4^{-2}$ ___

s) R R 17 = 17 ___

t) R $6^1 = \frac{1}{6}$ ___

u) R $6^2 = 6^{-2}$ ___

v) R $6^2 = \frac{1}{6^{-2}}$ ___

w) R $6^2 = \frac{1}{6^2}$ ___

x) $6^2 = \frac{1}{6^{-2}}$ ___

y) R $\frac{1}{8^3} = 8^3$ ___

z) $8^3 = 8^{-3}$ ___

Check answers to r-z now.

NEGATIVE EXPONENTS INVESTIGATED

*Unit 10

NOTE: Example kk) and ll) are definitions of division and subtraction. The definitions are based on multiplication and addition and the idea of opposites. This means, in theory, we don't have to divide. Instead, just multiply by the reciprocal. Same idea for subtraction - we could add the opposite and often do.

aa) $8^3 = \frac{1}{8^3}$ ___ bb) $8^3 = \frac{1}{8^{-3}}$ ___ cc) $8^{-3} = \frac{1}{8^3}$ ___

dd) $10^5 = \frac{1}{10^{-5}}$ ___ ee) R $\frac{2}{3} = \frac{3}{2}$ ___ ff) R $\frac{7^2}{7^3} = \frac{7^3}{7^2}$ ___

Let N, P represent any numbers but 0.

gg) R $N^{-1} = \frac{1}{N}$ ___

hh) R $N = N^{-1}$ ___

ii) $N \times N^{-1} = N^0$ ___

jj) $N^0 = 1$ ___

Reminders. If a x b = 1, then a is the reciprocal of b, and b of a.
So a=1/b and b=1/a a^{-1}=1/a a=1/a^{-1}
a^2 x a^{-2}= a^0= 1 R 6≠1/6^{-1} R6 =1/6=6^{-1}
R 2/3=1/2/3= 3/2= 1½

kk) $N^{-1} = \frac{1}{N}$ ___

ll) $N = \frac{1}{N^{-1}}$ ___

mm) $P - N = P + (-N)$ ___

nn) $P \div N = P \times \frac{1}{N}$ ___

oo) O_pO_p . . . $O_p O_p$ 5 = -5 ___

466 O_p's altogether

pp) R $\frac{2}{3} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$ ___

qq) $\frac{1}{\frac{4}{3}} = \frac{3}{4}$ ___

rr) $\frac{1}{\frac{3}{5}} = \frac{5}{3}$ ___

ss) $\underbrace{\frac{1}{R\frac{4}{5}} = \frac{1}{\frac{5}{4}}} = \frac{4}{5}$ ___

NEGATIVE EXPONENTS INVESTIGATED *Unit 10

26a) (8 x 8 x 8) x (8 x 8 x 8 x 8) = $\blacksquare$

8^3 x 8^4 = $8^{\square}$

b) Use calculator to find 8^7 ____________

c) Without calculator find 2^7. _____

d) $63.2 \div 10^3 = 63.2 \times 10^{-3}$ = ______

e) 56×2^{-3} = ____ f) 147×7^{-1} = ___ g) 200×5^{-1} = ____

h) $8^5 \div 8^3 = \frac{8^5}{8^3} = \frac{\overset{1x1x1}{\cancel{8}x\cancel{8}x\cancel{8}x8x8}}{\underset{1x1x1}{\cancel{8}x\cancel{8}x\cancel{8}}} = 8^{\square}$ = ____

> Note on so-called "canceling": The <u>same number</u> can divide the top and bottom of a fraction. $\frac{6 \div 2}{8 \div 2} = \frac{3}{4}$. In exercise 26h, 8 was divided three times into top and bottom. This works only if all numbers in the numerator are multiplied and all in the denominator also. Any addition or subtraction would mess it up.

i) $\frac{8^{28}}{8^{25}} = 8^{\square}$ = ____

> Mentally, picture 28 eights over 25 eights arranged as in h).

j) $\blacksquare \div 8^3 = \blacksquare \times 8^{-3}$. The same number must be put in each box (but not now).

Give one number which you are sure will work. _____

Give four other numbers which will work. ______, ______, ______, ______

k) $8^3 \div 8^5 = \frac{\overset{1x1x1}{\cancel{8}x\cancel{8}x\cancel{8}}}{\underset{1x1x1}{\cancel{8}x\cancel{8}x\cancel{8}x8x8}} = \frac{1}{8x8} = \frac{1}{8^{\square}}$ = _____

* l) $\frac{8^{25}}{8^{28}} = 8^{----} = \frac{1}{8^{\square}}$ = ___________

m) $8^2 \times \frac{1}{8^2}$ = __

n) $8^2 \times 8^{-2}$ = __

o) Therefore, $8^{-2} = \frac{1}{8^{\square}}$ and $8^2 = \frac{1}{8^{\square}}$

On your calculator, enter 2 and then the $\boxed{1/x}$ key , sometimes called the $\boxed{x^{-1}}$ key.
Try a few other numbers to be sure that you have the reciprocal key.

NEGATIVE EXPONENTS INVESTIGATED *Unit 10

Answers

1a) T **b)** 101.25; 101,250 10^5, 101,250 **c)** Addition

d) 2 **e)** .0003 or .000300 **f)** .00310 or .0031

2a) ^{4} **b)** ^{6} **3a)** ^{1} **b)** ^{3}

4) 3,620 **5)** ^{3} **6)** 1632.8 **7)** 16328

8a) NF, but $^{-3}$ is correct **b)** ^{4}

9) ^{5} **10a)** 320,000 **b)** .0032

11a)386 **b)**Yes! $38.6 \div (10^{50} \times 10^{51}) = 38.6 \div 10^{101}$,
a very tiny number; but $(38.6 \div 10^{50}) \times 10^{51} = 386$

12) NF, but $^{-3}$ is correct **13)** ^{9}

14) *Note:* Six hundred ten-thousandths is **.**0600 . Answer = 6,000

15) *Note:* 610 thousandths = **.**610 Answer = 61,000

16a) 23,350 **b)** 2,335 **c)** 233.5 **d)** 23.35

e) 2.335 **f)** .2335 **g)** .02335 **h)** .02335

17a) ^{7} **b)** ^{6} **c)** ^{5} **d)** ^{4}

e) 1 **f)** ^{3} **g)** ^{2} **h)** ^{1}

i) ^{0} , 1 **j)** ^{0}, 1 *Answers to 18 & 19 on next page*

20a) ^{3} **b)** $^{-3}$ **c)** ^{0}

21a) -8 **b)** 28 **c)** -8

22a) 3, probably **b)** 12 **c)** 1842 **d)** .1842 **e)** 47,230

23a) 20 **b)** 20 **c)** 3347 **d)** 3347

e) 9 **f)** 9

24) Zero has no reciprocal. 1/0 is <u>not</u> the name of *any* number.

25a) T **b)** T **c)** T **d)** F **e)** T, and is true for any even number of – signs

f) T **g)** T **h)** F **i)** T **j)** T **k)** T **l)** T **m)** T

n) T **o)** T **p)** 10^1 or 10 **q)** 10^0 or 1 **r)** T **s)** T

t) T **u)** T **v)** F **w)** T **x)** T **y)** T **z)** F **aa)** F

bb) T **cc)** T **dd)** T **ee)** T **ff)** T **gg)** F **hh)** T **ii)** T

jj) T kk) T **ll)** T **mm)** T **nn)** T **oo)** F **pp)** T **qq)** T

rr) T **ss)** T

NEGATIVE EXPONENTS INVESTIGATED *Unit 10

26a) ^{7} **b)** 2,097,152 **c)** 128 **d)** .0632 **e)** 7 **f)** 21 **g)** 40
h) ^{2}, 64 **i)** ^{3}, 512 **j)** Any numbers **k)** ^{2} =1/64 or .015625
l) $^{-3, 3,}$ 1/512 or .0019531 **m)** 1 **n)** 1 **o)** $^{2, -2}$

18a) ^{3} **b)** $^{-3}$
19a) $^{-3}$ **b)** ^{3} **c)** ^{0}

RECOLLECTIONS AND EXTENSIONS 1 Unit 11

Use a whole number to express remainders, for example:

20 ÷ 8 = 2 R4

No calculator on this paper

1) Starting at zero, a +5 jumper and a +6 jumper both land on 30.

a) Is 30 the LCM of 5 and 6? ____

b) Including 1, 5 and 6, find the complete list of whole numbers which have 30 as a common multiple. ______________________________

2) A +6 jumper is jumping in the following way: 32, 38, 44, 50, …

These landing places are not 6n numbers.

a) They are called 6n + ___ numbers

b) Will this jumper hit 99? ____

c) What is the remainder when any 6n + 2 number is divided by 6? ____

*d) What is the first number after 100 hit by a +6 jumper when starting on 32? _____.

*e) First after 1,000 starting on 32? ________

f) In part d) we can see from its start that the +6 jumper is not going to land on 6n numbers**;** it will land on 6n + 2 numbers. In looking for 6n + 2 numbers, find them easily by dividing numbers close after 100 or 1000 by 6, seeing if the <u>remainder</u> is 2. Get it? 32 has a remainder of 2 when divided by 6. Having only to find the remainder without caring about the whole answer, somehow allows greater ease when dividing, especially when working mentally**:** 100,300,009 ÷ 8 gives what remainder?_____

*3a) So, 1000 is a 6n + ____ number. The next 6n number (after 1000) is _____ and the first 6n + 2 number after that is _____.

b) What is the remainder when 1,000,000,000 is divided by 6? Start to divide by hand or mentally, but write the remainder as soon as you know it. ______

c) So, 104 is not a 6n number, it is a 6n + ____ number.

d) What is the first 6n number greater than 1,000,000,000? __________________

*e) Is $10^9 + 1$ a 6n + 5 number? _____ f) $10^9 + 5$ is a 6n + ___ number?

*4) What are the first four 3n + 2 numbers following 1,000,000? _______________________

____

RECOLLECTIONS AND EXTENSIONS 1 — Unit 11

5a) $23 + (-8) =$ ____ b) $23 - (-8) =$ ____ c) $23 - (+8) =$ ____ d) $-23 - (+8)=$ ____

6) $.003 \div 10^2 =$ _________ b) $84.0 \div 10^{-4} =$ _________(Remember double opposite.)

7) Let R= "Reciprocal of", and let RR mean "Reciprocal of reciprocal of".

a) R 3 = ____ b) R $\frac{1}{3}$ = ___ c) RR $\frac{1}{3}$ = ___ *d) R $\frac{1}{\frac{1}{3}}$ =___ *e) R $\frac{1}{\frac{1}{\frac{1}{3}}}$ = ___

8) Complete the series using exponents.

$8^3 + 8^2 +$ ____ + ____ + ____ + ____ + ____

11	10	9	8	7	6	5	4	3	2	1
$2^7=$ 128	$2^6=$ 64	$2^5=$ 32	$2^4=$ 16	$2^3=$ 8	$2^2=$ 4	$2^1=$ 2	___= 1	___= ___	$2^{-2}=$ $\frac{1}{4}$	___= ___
0	0	1	0	0	1	0	0			

9) The above is extended from the unit "Powers of 2". First, put a large 1 in the bottom cell of column 3, a 0 in the bottom cell of column 2 and a large 1 in the final cell.

a) In cells 4 - 1, fill in the blanks according to the patterns in the previous cells. Use cell 2 as a guide. We'll call the bold dot a "basimal" point because we are not working in base ten (decimal).

b) As you did in the "Powers of 2" unit, add together the place values that have 1 in the bottom cell: 32 + 4 + . . . Of course this time you will have to add fractions, too.
Final answer_________

c) Using all this as a guide, change the binary number 1011.011_2 to base ten. Don't recopy the chart. Just refer to it so you will know what to add.___________

10) Use double opposites to do these mentally:

a) $100 - (-2) =$ ___ b) $100 \div \frac{1}{2} =$ ___ c) $-100 - (-2) =$ ___ d) $-100 - (2) =$ ___

e) $20 - (-18) =$ ___ f) $-20 - (-18) =$ ___ g) $\frac{2}{3} \div \frac{2}{3} =$ ___ h) $\frac{200}{501} \div \frac{100}{501} =$ ___

i) $10^{30} \div 10^{-30} =$ ____ j) $\frac{1}{6^2} \div \frac{2}{6^2} =$ ___ *k) $10^{20} \div 10^{30} =$ ____

Having identity elements and opposites allows mathematics to be a powerful tool.

RECOLLECTIONS AND EXTENSIONS 1 Unit 11

11a) 60 - (-1) = ________ b) 32 – (-5) = ______ c) 138 – (-8) = ___ d) -50 – (-60) = ______

12a) 26 + (-6) = ________b) 26 – (-10) = _____ c) 2 – (-6) = _____d) -20 – (-5) = ______

13a) –1000 + (-2) = _____ b) –1000 + (+3) = ____c) – 21 – (-21) = ________

14a) 12 ÷ 1/6 = ________ b) 60 ÷ 3/4 = ________c) 60 x 3/4 = _____d) 5/6 ÷ 5/6 = ________

15a) 300 ÷ 3/4 = _______ b) 300 ÷ 4/3 = ______ c) 2/3 ÷ 3/2 = ________ d) 1/3 ÷ 2/6 = ____

16a) $8^{120} \div 8^{120}$ = _______ b) $6^{-3} \div 6^{3}$ = ________ c) $5^{6} \times 5^{-6}$= ______d) $1632^{2} \times 1632^{-2}$ = ____

Answers

1a) Yes **b)** 1, 2, 3, 5, 6, 10, 15, 30

2a) 2 **b)** no **c)** 2 **d)** 104 **e)** 1,004 **f)** 1

3a) 4, 1,002, 1,004 **b)** 4 **c)** 2

d) 1,000,000,002 **e)** yes f) 3

4) 1,000,004, 1,000,007, 1,000,010, 1,000,013

5a) 15 **b)** 31 **c)** 15 **d)** -31

6a) .00003 **b)** 840,000

7a) 1/3 **b)** 3 **c)** 1/3 ***d)** 1/3 ***e)** 3

8) 8^{1} (or 8) + 8^{0} (or 1) + 8^{-1} + 8^{-2} + 8^{-3}

9a)

2^7= 128	2^6= 64	2^5= 32	2^4= 16	2^3= 8	2^2= 4	2^1= 2	2^0= 1	2^{-1}= $\frac{1}{2}$	2^{-2}= $\frac{1}{4}$	2^{-3}= $\frac{1}{8}$
0	0	1	0	1	1	0	0	1	0	1

RECOLLECTIONS AND EXTENSIONS 1 Unit 11

Answers *(cont.)*

9b) 36 5/8		**c)** 11 3/8			
10a) 102	**b)**200	**c)** –98	**d)** –102	**e)** 38	**f)** –2
g) 1	**h)** 2	**i)** 10^{60}	**j)** ½	**k)** 10^{-10} or $1/10^{10}$	
11a) 61	**b)** 37	**c)**146	**d)**10		
12a)20	**b)** 36	**c)** 8	**d)** -15		
13a) –1002		**b)** -997		**c)** 0	
14a) 72	**b)** 80	**c)** 45	**d)** 1		
15a) 400	**b)** 225	**c)** 4/9	**d)** 1		
16a) 1	**b)** 6^{-6} or 1/46656		**c)** 1	**d)** 1	

A GLIMPSE INTO THE INFINITE *Unit 12

All this is full, all that is full. From fullness, fullness comes.
When fullness is taken from fullness,
Fullness still remains. (A saying of ancient India, from the Upanishads)

This introductory poem might seem to make little sense, but consider this**:** when a devoted mother and father love their child fully, they will still love her/him just as fully when a second child is born. And no doubt the second child will be loved fully too. The fullness of love does not split down the middle. Light also works this way. A person in a fully lighted room does not receive less light when others enter (unless someone is very tall and casts a shadow). The light is not diminished in any way just by being "used". Outside, the sun's rays are always present during daylight hours, even though often diffused or scattered by clouds. The sun seems to provide us with an inexhaustible, or infinite, supply of light and is not in the least affected by how its light is being used, or how much is being used. Astronomers tell us, of course, that in millions of years our sun will change form and no longer provide light, so its supply of light is very, very great but not inexhaustible.

So, what *is* inexhaustible? What is truly infinite or never-ending? You are given a chance to express your opinion in the following examples. Finite means limited or not infinite and is pronounced fie-nite. Infinite is pronounced in-fin-it, long i's in finite and short i's in infinite. Though not infinite, finite may refer to something very, very large. Circle either "finite" or "infinite"**:**

1) The set of counting numbers {1, 2, 3, 4, etc.} Finite Infinite

2) The number of leaves in the whole world. Finite Infinite

3) The set of even counting numbers {2, 4, 6, etc}. Finite Infinite

*4) The number of possible rectangles of different shapes, each having an area of 12 square inches. Finite Infinite

5) The number of integers less than 100. (Integers include 0, 1, 2, 3, etc., and -1, -2, -3 etc.). Finite Infinite

6) The number of numerals expressible in base two. Finite Infinite

7) The number of molecules of water vapor in the earth's atmosphere. Finite Infinite

A GLIMPSE INTO THE INFINITE *Unit 12

8) A mathematician once said that the number of grains of sand on the beach at Coney Island is much less than: 10,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000, 000,000,000,000, 000,000,000,000,000,000,000,000,000,000,000,000,000 grains.

The mathematician's nephew, age 9, named the above number a googol, because "It has so many zeroes".

8a) 1 googol = $10^{\boxed{}}$ (Recall that 10,000,000 = 10^7)

8b) 1 googol is: __________ (Finite / Infinite)

*9) 1 googol is 1 with a hundred zeros after it. In the same way, describe a googolplex if 10^{googol} = 1 googolplex.

a) A googolplex is a 1 with how many zeros after it? ______________________

b) A googolplex is**:** (Finite/ Infinite)

Check your opinions and answers with those on the next page. Discussion is included.

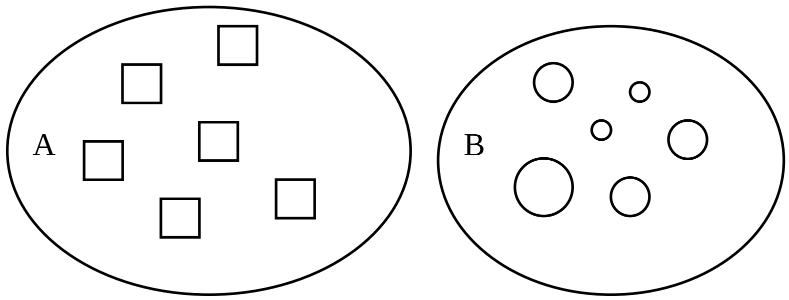
A small child sometimes asks, "What is the biggest number?". This is usually after she has learned something about counting and sees that numbers go farther and farther. But that child is not inclined to be satisfied with an answer like "They just keep on going—there is no end". The child might still insist "But what's the biggest?"

You have a sense of very large numbers such as a googol or even a googolplex and larger. It is helpful to view the infinite as non-ending, perhaps through a system or plan that is said to provide a continuous and non-ending procedure such as counting in base ten, or in base two or some other base number. Although nobody can actually carry out the procedure "unendingly", we are able to see that it can be spoken of and understood by another person as unending. That is, there is no last, or highest number. If anyone claims that there is, ask "What about that number + 1?" Perhaps the infinite exists only as an idea in the mind, suggesting something imagined, not real. (?)

Our view of an infinite set of objects as the result of an unending procedure does not serve as a definition of infinite sets. Mathematicians require something much more specific which would clearly rule *out* all sets not infinite, as well as rule *in* all sets infinite. Surprisingly, we need a kindergarten or grade 1 activity to help us arrive at such a definition.

A GLIMPSE INTO THE INFINITE *Unit 12

When you were a small child in school you might have been given two simple groups or sets of objects to match and tell which set had more (or less) or whether the sets had the same number of objects.



Here, the number of objects in Set A = the number in Set B, but as a kindergarten pupil perhaps you were required to *draw lines matching* the objects between Set A and Set B to be sure of **showing** the matching. If there is one member of Set A for each member of Set B **and** one member of Set B for each member of Set A (none left over in either set), then the child knows that the number of objects in each set is the same. *We* say that such sets have their members in 1 to 1 correspondence. Such sets are said to be *equivalent.* We do not say that those sets are equal. Equal means the same. Not merely the same number of objects but exactly the same objects. The sets {a, 3, 6} and {*vi*, a, 3} are equal, even though the names for the numbers are in different order and the six has different form.

In mathematics, C = D if and only if C and D are names for the same object. Sets A and B above are not the same set. Their *numbers* are equal. Because of this the sets themselves are only equ*ivalent*. We do not confuse small children with this distinction.

Let's see what the simple idea of 1 to 1 correspondence has to do with infinite sets. The set N below is the set of natural numbers. The set E can be thought of as set N with the set of odd numbers removed, leaving the set of even numbers.

$$
\begin{array}{llllll}
N = \{1, & 2, & 3, & 4, & 5, & 6 \;\ldots\} \\
\quad\;\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
E = \{2, & 4, & 6, & 8, & 10, & 12 \ldots\}
\end{array}
$$

10) What number in set E corresponds with 10 in set N? ______________________

A GLIMPSE INTO THE INFINITE *Unit 12

11) What number in set E corresponds with 140,000 in set N? ________________

12) What number in set N corresponds with 140,000 in set E? ________________

13) Is there a number in N which corresponds with 1 googol in E ? (Yes/no) ______

14) Is googol + 1 in set N? ____ 15) Is googol + 1 in set E? ____

16) Can we pick a number in E and be sure that it corresponds to some number in N? ___

17) Can we pick a number in N and be sure that it corresponds to some number in E? ___

18) Are set N and set E in 1 to 1 correspondence? ______

19) Below, show a 1 to 1 correspondence between the natural numbers, set N, and the odd numbers, set O, as was done above with sets N and E. Start with the first odd number and don't forget brackets and dots.

$$
\begin{array}{lccccccc}
E = \{2, & 4, & 6, & 8, & 10, & 12\ldots\} & & \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\
N = \{1, & 2, & 3, & 4, & 5, & 6, & 7, & \ldots\} \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
O = \{1, & & & & & & &
\end{array}
$$

A few more exercises: These are simple but perhaps elusive at first.

20) What number in set O corresponds with 8 in set N? _____

21) What number in set O corresponds with 100 in set N? _____

22) What number in set N corresponds with 999 in set O? _____

23) Are sets N and O in 1 to 1 correspondence? (Yes/No)? _____

24) What number in set E (yes, E) corresponds with:

a) 1 in set O? _____ b) 13 in set O? _____ c) 999 in set O? _____

25) What number in set O corresponds with:

a) 2 in set E? _____ b) 50,000 in set E? ___________

26) Are sets E and O in 1 to 1 correspondence? ______

In the late 1800's, during the time of the German mathematician Georg Cantor, the infinite in general was viewed with great skepticism. The idea that anything, like the natural numbers, could be viewed as unending and at the same time be grouped and named as if it were a completed thing, was unacceptable to many in the field of mathematics.

But Cantor, undaunted, went on largely by himself, to do ingenious, original work on the infinite. Morris Kline, in his book *Mathematics in Western Culture*, says that "Cantor's greatness lies in his perception of the importance of the one-to-one correspondence principle and in his courage to pursue its consequences". See our Book List #12.

Cantor gave a name to the *number of members* in the set N, of natural numbers. The name was $\aleph_0$, called aleph null, with a long "a". In other words, the question "How many natural numbers are there?" is answered by saying "Aleph Null". Aleph ($\aleph$) is the name of the first letter of the Hebrew alphabet and null is nothing, or no thing. This suggests that a *first* or *lowest possible* infinite value is meant by $\aleph_0$.

Amazement #1: The notion of a first or lowest infinity suggests that there are larger infinities than the number of counting numbers. We consider this more closely in a later unit.

Amazement #2 (Definition): Any set is infinite if, and only if, the set is in 1 to 1 correspondence with a *part* of itself. This can be stated as "A set is infinite if, and only if, the set is in one-to-one correspondence with a proper subset of itself". The word "proper" in this case means that at least one member of the original set is missing from the subset. Most people would think that about any subset but it is not so.

Clearly, any finite set such as {1, 2, 3, 4} has several proper subsets:

> {1, 2, 3}, {1, 3}, {2}, and lots more. None of these will be in 1 to 1 correspondence with {1, 2, 3, 4} and so do not fit the definition in Amazement #2. {1, 2, 3, 4} is a subset, but not a *proper* subset, of {1, 2, 3, 4}.

Remember that sets in 1 to 1 correspondence are equivalent. They each have the same number of elements (as in kindergarten).

27a) Since set E and set N are in 1 to 1 correspondence, how many even numbers are in the infinite set E? _____

b) How many odd numbers in set O? _____

(Note: There is a specific answer to Exercise 27 on this page. "Infinite" is not the specific answer.)

A GLIMPSE INTO THE INFINITE *Unit 12

Amazement #3: Each of the even number and odd number sets has the same number of members as the set of natural numbers. The answers to 27a and 27b are $\aleph_0$ and $\aleph_0$. Here is something else amazing: How much is $\aleph_0 + \aleph_0$? You can figure this out by thinking of addition in the usual way of "putting together" set O and set E:

$$O = \{1, \quad 3, \quad 5, \quad 7, \ldots\}$$

$$E = \{2, \quad 4, \quad 6, \quad 8, \ldots\}$$

$$\text{Result: } N = \{1, 2, 3, 4, 5, 6, 7, 8 \ldots\}$$

The number of even numbers + the number of odd numbers =
the number of natural numbers, thus**:**

Amazement #4: $\aleph_0 + \aleph_0 = \aleph_0$**!** No finite numbers behave this way (except zero). Of course this example is not a demonstration for *all* transfinite numbers, only for odd and even whole numbers. (Transfinite is Cantor's name for infinite.) Now, again, please consider the introductory verse about fullness, especially the last two lines: "When fullness is taken from fullness, fullness still remains." Think of $\aleph_0$ (Aleph null) as fullness because it is quite well understood that they do not end and are therefore infinite in number.

In the sets pictured at the top of this page O, E and N, each has $\aleph_0$ members and therefore each can be thought of as fullness. The picture can be thought of as subtraction as well as addition: Take E from N, and O is left. Thus,

Amazement # 5: $\aleph_0 - \aleph_0 = \aleph_0$, but not always. To extract O **and** E from N leaves nothing. To extract O **or** E (but not both) leaves an infinite set with $\aleph_0$ members. To extract other infinite sets such as squares, cubes or primes would leave $\aleph_0$ members. Later, you will see some astonishing feats enabled by one-to-one correspondence

A GLIMPSE INTO THE INFINITE *Unit 12

Answers

1) Infinite

2) Number of leaves in the world**:** Large but finite

3) Infinite

4) Infinite. Rectangles with 12 square inch areas are 3 x 4, 2 x 6, 1 x 12, but also 1/2 x 24, 1/3 x 36, 1/10 x 120, 1/100 x 2400, 1/2,000 x 24,000 etc., etc.

5) Infinite (in the negative direction).

6) Infinite. Place value allows the visualizing of numerals without a stopping place.

7) Very large, but finite.

8a) The missing exponent is 100 **8b)** Finite

***9a)** A one with <u>a googol of zeros after it</u>. That is many billions of zeros. **9b)** Finite Return to page 2

10) 20 **11)** 280,000 **12)** 70,000

13) Yes, because 2 x 1 googol would end in zero, and is therefore even, putting it in E.

14) Yes **15)** No (Googol is even, so googol + 1 is odd.).

16) Yes **17)** Yes **18)** Yes

19)

$$\begin{array}{cccccccccc} N = \{1, & 2, & 3, & 4, & 5, & 6, & 7, & \ldots\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ O = \{1, & 3, & 5, & 7, & 9, & 11, & 13, & \ldots\} \end{array}$$

20) 15 **21)** 199 **22)** 500 **23)** Yes

24a) 2 **b)**14 **c)** 1,000

25a) 1 **b)** 49,999

26) Yes

27a) and **b)** $\aleph_0$ and $\aleph_0$

DECEPTIVE RATES **Unit 13

Here is a problem which looks fairly easy but is not:

Kevin can polish his mother's car in 2 hours but it takes his younger brother Jed 3 hours to do the same job. Working together, how long will it take them to do the job? Assume that working together does not change their working rates.

What do you think is the most likely wrong answer to this problem? _______

If you gave 5 hours as the likely wrong answer, you were correct. It certainly is not going to take both of them together longer than one of them alone. Of course there are many other wrong answers and you might consider some of them quite likely, like 2 ½ hours - - halfway between their two rates. But that is still not less time than Kevin alone requires.

Without a good knowledge of algebra, most people would be quite baffled by this problem. Perhaps *you* are. And it is probable that many algebra students go through the rest of their lives knowing *how* to do the problem but not knowing *why* the equations they were taught, work. It is clear that the answer must be somewhat less than 2 hours, because Kevin alone can wash the car in 2 hours. You *know* this is true, but where do we go from here?

Detour:

1) T/F: A walking rate of 2 miles per hour is also 1/2 hour per mile. ____

These rates are backwards from each other. One is miles per hour and the other is hours per mile.

Simple but important!

The miles per hour rate is 2 miles per hour, but its opposite, the hours per mile rate is ½ hour per mile. In unit 10 we called these reciprocals (½ x 2 = 1).

2) Polishing 3/5 car per hour, when turned the other way, is ____ hours per entire car.

(Ans. 5/3 or 1 2/3 hours per car)

3) Crawling at the rate of ¾ miles per hour is the rate of ___ hours per mile.

4) Taking 3 hours to paint a shed is a rate of ____ shed per hour.

DECEPTIVE RATES **Unit 13

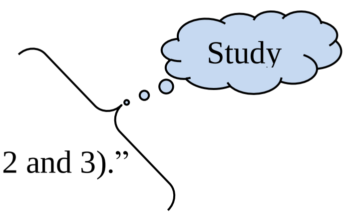
Jumping rope might be timed in jumps per minute but parachute jumping from a flying plane would be timed better in minutes per jump. We tend to use rates which are greater than 1, like 4 miles per hour, rather than ¼ hour per mile, even though either way is mathematically correct. Now we can return to the original problem from the beginning of this unit:

Jed's rate is 3 hours per car and Kevin's is 2 hours per car.

5) Change each rate to car(s) per hour. Jed's rate: ____ car(s) per hour.
Kevin's rate: ____ car(s) per hour.

6) How many cars, (or how much of a car) do they polish working together? ___ car(s) per hour.

7) Knowing their combined car(s) per hour rate is 5/6 cars per hour, we can now see that their combined hours per car rate is _____ (hours per car). This answers the original problem, 6/5 hours, or 1 1/5 hours, or 1 hour and 12 minutes.



We can get a different handle on all of this by finding:

"The reciprocal of (the sum of the reciprocals of 2 and 3)."

$$\underbrace{\frac{1}{\underbrace{\frac{1}{2}+\frac{1}{3}}_{\text{the sum of the reciprocals of 2 and 3}}}}_{\text{the reciprocal of}} = \frac{1}{5/6} = \text{6/5 or 1 1/5 hours, or 1 hour 12 minutes.}$$

8) Find the reciprocal of the sum of the reciprocals:

a) 3 and 6 b) 1/2 and 4 c) 10 and 15 d) 1.5 and 2.5

9) On an old fashioned bathtub the hot water faucet alone can fill the tub to the drain overflow in 12 minutes. The cold water faucet can do it in 9 minutes. How long will it take with both faucets open?____________

10) Harriet can wash her mother's car in 15 minutes. Her older brother Michael takes 10 minutes. However, their little brother Paine's help reduces their respective rates by 20%. Paine can (theoretically) do the job alone in 72 minutes. Note: Paine's "help" does not *reduce* Harriet's and Michael's time. How long does this family fiasco take? _________

DECEPTIVE RATES **Unit 13

Answers

1) True

2) 5/3 or 1 2/3 hrs.

3) 4/3

4) 1/3

5) 1/3, 1/2

6) 5/6

7) 6/5 or 1 1/5

8a) 2 **b)** 4/9 **c)** 6 **d)** 15/16

9) 5 1/7 min.

For exercise 10 you probably had at one point: Note that the lowest common denominator of these fractions is 72.

$$\frac{1}{\frac{1}{72}+\frac{1}{12}+\frac{1}{18}}$$

This enables you to rewrite the above expression as:

$$\frac{1}{\frac{1+6+4}{72}}$$

Final answer for **exercise 10:** 72/11 or 6 6/11

TIDAS AND HIS CROWS

Unit 14

Long ago and far away there lived a mighty king named Tidas. Tidas ruled a happy and peaceful kingdom and his subjects loved him dearly for his great kindness. Sometimes however, he acted a little strangely and did things which made some people wonder whether he was quite all right. This really didn't bother his subjects, however. They usually smiled knowingly among themselves when they heard of Tidas' latest adventure and found it easy to forgive him for his odd ways since he was such a kind old king.

One quiet day in June, Tidas was relaxing on the veranda in his white Bermuda shorts sipping cool pink lemonade (one of his favorite pastimes) and idly watching some crows steal the seeds from his sunflowers (one of <u>their</u> favorite pastimes). Egbert, Tidas' manservant, was gently fanning his king with a large fern leaf and deftly brushing away any flies which dared to alight upon his majesty's royal hide. Egbert had been a faithful and loyal manservant to his king for many years and Tidas often discussed important matters of state with him. Those close to the king whispered that it was Egbert who kept the king on a steady course by cleverly guiding the many decisions that it takes to keep a kingdom happy and peaceful.

Tidas shifted slightly on his soft velvet couch.

"Egbert." he said lazily.

"Yes, O Mighty King," replied Egbert.

"Egbert, why do you suppose all crows are black?" said Tidas, sounding as if he didn't really care and was just asking to make conversation.

"That is a very clever question, Your Majesty," said Egbert respectfully.

"Really?" said Tidas, beginning to sit up and take notice. "Why?" (Even a wise king likes compliments.)

"To give you a reason, Sire, would suggest that I agree that all crows are black."

Tidas looked puzzled. "Do you mean all crows <u>are</u> not black?"

"Oh no, Sire," answered Egbert.

By this time, Tidas was thoroughly confused and upset. "Stop waving that silly fern at me and make sense. First you say that you cannot agree that all crows <u>are</u> black and now you cannot agree that all crows are <u>not</u> black. Now which is it?"

"Both, Your Majesty," replied Egbert calmly.

Upon hearing this, King Tidas rose from his couch, spilling pink lemonade on his white Bermuda shorts.

"Egbert!" he stormed. "You are trying to make a fool of me."

"Please, O Wise One," said Egbert. "Be seated and I will explain."

Slowly, and still somewhat peeved, Tidas settled back on his couch.

"Sire, to state that not all crows are black is to state that there is at least one crow which is some color other than black. Since I have never heard of such a crow, I cannot agree with the statement that not all crows are black." (Notice that Egbert has improved the position of the word "not" in Tidas' unclear statement "All crows are not black".)

"Of course," said Tidas, beginning to brighten a little. "That is certainly clear, but what about the other statement?"

"Ah yes," said Egbert, "To state that all crows <u>are</u> black is to state that you have a definite knowledge about the color of every crow in the world. I have not observed every crow in the world and I know of no one who has. Therefore, I cannot agree that all crows are black."

"Well," said King Tidas slowly, "I—I begin to see your point. But you and I have observed many, many crows from my veranda. Don't you think that it's safe to say that all crows are black?"

"Ah, Your Majesty. Now you are making a generalization."

"General who?" asked Tidas. "Why drag the army into this?"

"No, no, Sire," replied Egbert quickly. "A generalization formed by induction, is a statement which is made about <u>all</u> of the particular things you are talking about. Take our crows, for instance. You are making a generalization about <u>all</u> crows when you have observed only a rather small part of all crows. This is called an inductive leap and in this case the leap might seem rather large."

"Rubbish!" snorted Tidas. "This is nonsense. I <u>know</u> that all crows are black and if you had any sense you would admit that you know it too, Egbert."

"We must be careful, Your Majesty," said Egbert, "not to fall into the trap of generalizing too quickly, like the anthropologist who concluded that all Indians walk single file because the only Indian he ever saw was walking that way."

TIDAS AND HIS CROWS Unit 14

"Ha, ha," laughed Tidas. "Good one, Egbert, very good. That reminds me of the 9th century scholar who argued that the moon is superior to the sun because the moon gives us light at night, but the sun comes out only in the daytime when we don't need its light."

Egbert continued without comment: "The type of reasoning we are talking about, Your Majesty, is called inductive reasoning, or simply, induction. We observe as many individual cases as possible and then make our generalization. As we said, there is always some danger involved in making the leap from the cases we have actually observed to <u>all</u> cases. This leap is called the inductive leap and should be kept as small as possible.

"Egbert," said King Tidas, "You have convinced me. We must be scientific about this." He stood up and began pacing up and down the veranda. "I will send bird watchers to all parts of the world to observe crows. They will watch and ask everyone they meet if they have ever seen a crow which is not black."

"But Sire," protested Egbert. "Do you really think this is nec---?"

"Enough!" interrupted Tidas. "I have made up my mind. I will go now and assemble three hundred bird watchers and give them their instructions---that is,---after I change my shorts. This lemonade is getting sticky."

Egbert sighed as he watched his king stride away from the veranda in the direction of his chambers. He knew that there was no point in arguing with him in his present state of mind. He could only hope that Tidas would forget about his plan to experiment in inductive reasoning by the time he finished changing his shorts.

But Tidas did not forget. He sent out his watchers as he said he would, with instructions to observe and inquire about crows which are not black. Needless to say, the watchers were quite mystified about their instructions but they were wise enough not to risk the wrath of their ruler by questioning his orders.

Several months passed. The time drew near for the watchers to return and report to King Tidas. Tidas summoned Egbert to discuss with him the possible outcomes of the experiment.

"Egbert," he said, "suppose that all three hundred of my watchers report that they have neither seen nor heard of any crows other than black ones. Would you then think it would be safe to say that all crows are black?" he asked, watching Egbert hopefully.

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"Yes, Sire," he said. "I believe that this would be a valid conclusion."

The day appointed for the watchers' return arrived. One by one they were led into the king's private chambers where they were required to kneel and whisper their report into the king's ear as he sat in one of his large, carved chamber chairs. With each report Tidas would nod, smile with satisfaction, and give each watcher a pouch of gold.

"Now," proclaimed Tidas, "I desire that all my subjects within a hundred miles who are able, should assemble in the square beneath my balcony, at noon, three days hence."

Word went out. On the morning of the day designated, the square began to fill with people.

"What's Tidas got on his mind this time?" someone asked.

"I hope he isn't going to make us all wear Bermuda shorts to celebrate his birthday the way he did last year," spoke another. "I have boney knees."

The hands of the large clock in the spire rising behind Tidas' balcony drew near twelve and the people waited in the square with hushed expectancy.

Precisely at the first strike of twelve, King Tidas emerged from his balcony doors dressed in long flowing robes and jeweled crown. The crowd cheered with enthusiasm revealing their true affection for an eccentric but devoted leader.

The clamor died, the crowd waited, respectful. Tidas raised his arm and his voice soared over the crowd. "I, King Tidas, have made a Royal Generalization and do hereby decree that - - ALL CROWS ARE BLACK!"

Silence followed. Tidas lowered his arm, turned, and walked slowly back through the balcony doorway. The people stood with mouths open, staring at the empty balcony.

Gradually they began to find their tongues.

"Well, Tidas has done it again."

"What a remarkable thing to say. 'All crows are black.'"

"As if we didn't know!"

Slowly, the people began to drift away, shaking their heads. A few were angry or in poor humor but most smiled good-naturedly and began to chat among themselves. They remembered the days when their kingdom was ruled by cruel tyrants. They were grateful for the kindness of

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Tidas and were quite willing to go along with his whims for the sake of peace and happiness. “Long live King Tidas” were among murmurs heard. “Let him remain as he is.”

Although induction could not be accepted as *proof* in mathematics (it is more like an educated guess) it is used frequently.

The following addition examples are written horizontally. Add them and write the answers to the right. Then go back, and to the left of each example, write the number of terms (numbers) in the example. Do not count the answer.

___ 1 = 1

___ 1 +3 = ___

___ 1 + 3 + 5 = ___

___ 1 + 3 + 5 + 7 = ___

___ 1 + 3 + 5 + 7 + 9 = ___

Now go back and see if you can find any relationship between the answer to the example and the number of terms in it.

1) The sum of the first 3 odd numbers is _______.
(Remember, this is not 1 + 3)

2) The sum of the first 8 odd numbers is _______

3) The sum of the first 50 odd numbers is _______

In order to do example 3 without *writing down* the first 50 odd numbers and adding them, you had to discover the generalization which is suggested by the sums. You discovered the generalization by induction. You saw that all that needs doing is to square the number of terms.

Another way of saying this is, if n stands for *any* number of terms, the sum of the first 4 odd numbers is n^2. For example the first 4 odd numbers is 4^2, or 16. The sum of the first 10 odd numbers is 10^2, or 100.

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What is the sum of the first n odd numbers when:

4) n is 92? _______

5) n is 40? _______

6) n is 6? _______ Verify this by writing down the (that is, show that this is true) by writing down the first 6 odd numbers and adding them.

7) What is the sum of the first 160,000 odd numbers? _________

In mathematics, the type if induction involving an inductive leap is a useful tool for discovering generalizations. However, inductive reasoning of this type cannot be accepted as *proof* in mathematics since it is not usually possible to test <u>all</u> cases. If you <u>can</u> test all cases in some non-infinite (finite) situation, this is called perfect induction and *is* a proof. Another type of induction, usually studied in second year algebra, is called Mathematical Induction, but this requires rather advanced algebra knowledge so you are spared from that here.

1 $1= 1$

2 $1 +3 = 4$

3 $1 + 3 + 5 = 9$

4 $1 + 3 + 5 + 7 = 16$

5 $1 + 3 + 5 + 7 + 9 = 25$

Answers

1) 9 **2)** 64 **3)** 2500 **4)** 8464 **5)** 1600 **6)** 36 **7)** 25,600,000,000 or 256×10^8

SERIES *Unit 15

Earlier, we found a general rule for the sum the first n odd numbers. The rule was to square the number of terms. Thus,

1. 1	= 1	1. 2	= 2
2. 1 + 3	= 4	2. 2 + 4	= ___
3. 1 + 3 + 5	= 9	3. 2 + 4 + 6	= ___
4. 1 + 3 + 5 + 7	=16	4. ____________	= ___

Try evens.

1) If we write the answers for odd numbers horizontally, we notice that they form a series. Write the next 5 terms of the sequence: 1, 4, 9, 16, ____, ____, ____, ____, ____.

2) What would be the 70th term of this sequence? ______

3) What is the 300th term? __________

Often we can discover the secret for extending a series either by using the number of the term or by seeing what was done to one term to get the next.

Answers to "Try evens" at top.

1. 2 = 2
2. 2 + 4 = 6
3. 2 + 4 + 6 = 12
4. 2 + 4 + 6 + 8 = 20, etc.

There is a general rule for getting the sum of the first n *even* numbers.
If you want a real brain teaser, try to find a rule before reading further.

Complete:

4) Number of terms (n) $\longrightarrow$ 1 2 3 4 5 6 7 ___ ___ ___ ___ ___

5) Sums of first n odd numbers (n^2) $\longrightarrow$ 1 4 9 16 25 36 49 ___ ___ ____ ____ ____

6) Sums of first n even numbers $\xrightarrow{(n^2 + n)}$ 2 6 12 20 30 42 56 ____ ____ ____ ____ ____

Does any relationship stand out among the sequences above? Look down the columns as well as across the rows. Apparently, looking at exercises 4, 5, 6, to get the sum of the first n even numbers, add the number of the term n, to the sum of the first n odd numbers.

7) The sum of the first ten even numbers is ______

8) The sum of the first twelve odd numbers is ______

9) The sum of thee first nine even numbers is ______

Using a chart is a little bulky and not at all practical for large values. Look back at the chart to see the sum of the first n numbers. We added the number of the term (n) , to the sum of the first n odd numbers (n^2). So the formula is:

$$S = n^2 + n \text{ for the sum of the first n even numbers.}$$

Example: Find the sum of the first 16 even numbers.

Solution: since n = 16, and $S = n + n^2$, then

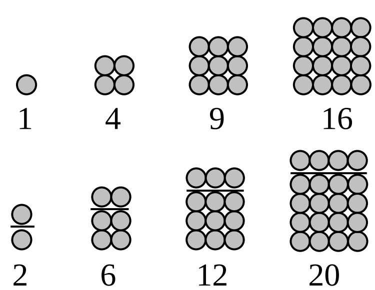
$$S = 16^2 + 16$$
$$S = 256 + 16$$
$$S = 272$$

Use the appropriate formula to find the sum of the first:

10) 10 even numbers. $S = n^2 + n = 10^2 + 10 =$ ____ + 10 = _____

11) 24 even numbers. S = _____

12) 16 odd numbers. $S = n^2 =$ ____



Sometimes pictures or diagrams are helpful in dealing with formulas. Those at left are diagrams of the sums of the first 4 *odd* numbers (n^2).

Again at left are diagrams for the sums of the first 4 *even* numbers ($n^2 + n$). Each has n circles added to the top of the n^2 part of the diagram. Study this for understanding.

Show *diagrams* to find the sum of: (Use dots rather than circles if you want to.)

13) The first 6 odd numbers: S = _____

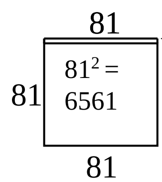
14) The first 6 even numbers: S = _____

15) The first 7 even numbers: S = _____

Show diagrams for 13-15 here.

16) The first 81 even numbers S = _____

(Instead of dots or circles, we can draw a geometric figure.)



This line segment is actually a *rectangle* with length 81 units and height 1 unit, almost too skinny to show up.

6561 + 81 = _______

*17) A rich man said to his son, "I will deposit 2 cents in the bank today, 4 cents tomorrow, 6 cents the next day, and so forth."

a) How much money will be in the account after 20 deposits? _______

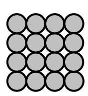
b) How much after 1000 deposits? _____________

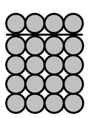
Show diagrams to find the following sums: (See exercise 16.)

18) The first 150 odd numbers

19) The first 150 even numbers

20) The first 57 even numbers.

21a)  This is the picture which describes the sum of the first 4 _______ numbers.
(odd/even)

b)  This diagram describes the answer to:
"What is the sum of the first ______ _________ numbers?"

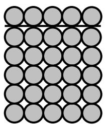
*22) Likewise, 110 is the answer to "What is the sum of the first ___ _______ numbers?"

23) 20 is the sum of the first ____ even numbers? You might think "What number squared, plus the number, equals 20" (For what number n, does $n^2 + n = 20$?). <u>But,</u> notice that the number of circles above is ___ x ___ = 20, because there are <u>5 rows of 4</u>.

*24) Use this idea to find the number of rows and number of circles in each row when the sum of the first n even numbers is 240. No. of rows = _____ No. in each row = _____

25) Use a calculator to help you find the number of rows and the number of circles in each row for the following sums of *even* numbers starting with 2:

a) 56 = ___ x ___ *b) 132 = ___ x ___ *c) 420 = ____ x ____

26)  We have been regarding this diagram as having $5^2 + 5 = 30$ circles. Or, we can ignore the line that separates the 5^2 circles from the 5 circles, and then find the total number of circles by simply multiplying ___ x ___.

27) If asked to find what two consecutive number you would multiply to give 756, you could do it by trial and error. But, there is a single key on the calculator which, when applied to 756, would give you a close approximation to one of the two numbers. Discover the key, check, and tell the two consecutive numbers. _______________

28) For each of the following quantities n, tell how many of the first n *even* numbers are being added?

a) 110 ____ b) 2862 _______ c) 209306 __________

29)

Number of terms (n) →	1	2	3	4	5	6	7 . . .							
Sum of first n odd numbers (n^2) →	1	4	9	16	25	36	49 . . .							
Sum of first n even numbers $\xrightarrow{(n^2 + n)}$	2	6	12	20	30	42	56	____	____	____	____	____		
	1x2	2x3	3x4	4x5	____	____	____	____	____	____	____	____		

Fill in these blanks.

30) Using the chart above, tell the 15th term in each of the two bottom rows. ____, _____

31) (T/F) The product of two consecutive whole numbers is always even. ____

Answers

1) . . .25, 36, 49, 64, 81

2) 4900

3) 90,000

4) . . . 8 9 10 11 12

5) . . . 64 81 100 121 144

6) . . . 72 90 110 132 156

7) 110

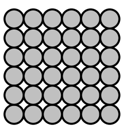
8) 144

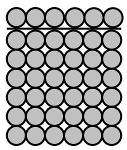
9) 90

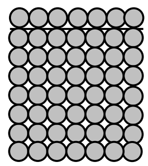
10) 100, 110

11) 600

12) 256

13) 36 

14) 42 

15) 56 

16) 6642

17a) \$4.20 **b)** \$10,010

18) 22,500

19) 22,650

20) 3,306

21a) odd **b)** four even

22) ten even

23) 4, 5x4 or 4x5

24) 16, 15

25a) 7x8 or 8x7 **b)** 11x12 or 12x11 **c)** 20x21 or 21x20

26) 5 x 6

27) $\sqrt{\ }$ key, 27 x 28

28a) 10 **b)** 53 **c)** 457

29)

			72	90	110	132	156
5x6	6x7	7x8	8x9	9x10	10x11	11x12	12x13

30) 240, 15x16

31) T

MORE SERIES **Unit 16

The sum of the first n consecutive natural numbers:

1 = 1

1 + 2 = 3

1 + 2 + 3 = 6

1 + 2 + 3 + 4 = 10 etc.

1) Arranging the answers in a sequence we get: 1, 3, 6, 10, 15, ___, ___, ___, ___,

Write the next four numbers in the series.

Here is the chart we worked with earlier. Finish the whole chart.

2) Number of terms (n)	1	2	3	4	5	6	__	__	__	___
3) Sums of first n odd numbers (n^2)	1	4	9	16	25	36	___	___	___	___
4) Sums of first n even numbers ($n^2 + n$)	2	6	12	20	30	42	___	___	___	___
5) Sums of first n consecutive natural numbers, the so-called <u>triangular numbers</u>.	1	3	6	10	15	21	___	___	___	___

*6) Observe how the row of triangular numbers is related to the row above it and give a formula for the sum of the first n natural numbers. S =________________________

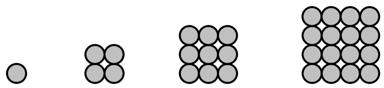
7) What is the sum of the first 8 natural numbers? _____

8) Find the sum of the first 100 even numbers. ___________________ (Use exercise 6)

9) 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = ____

Recall that the second row of numbers (Exercise 3) could be represented this way:

MORE SERIES **Unit 16



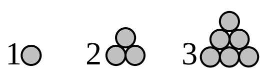
1 4 9 16 etc.

It is not coincidence that these *numbers* are called "squares".

10) T/F: You noticed (we hope) that the numbers for exercise 5 were exactly half of the numbers for exercise 4, above. This accounts for the ½ in the formula (or the 2 in the denominator) for the sum of the first n natural numbers. _____

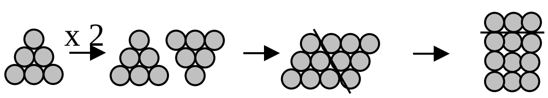
11) Carefully draw "triangles" 4 and 5 of the triangular numbers in the sequence below and tell how many little circles are in each.

Note how the new base line fits neatly under the old base and is one larger.

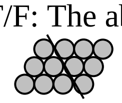


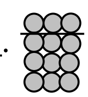
4 _______ 5 ________

1 3 6 ___ ____

12) 

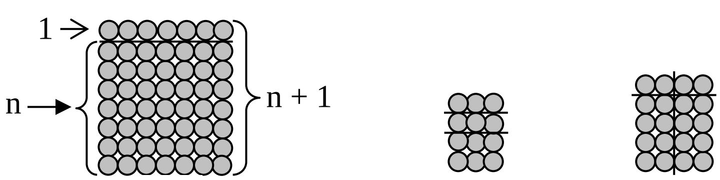
a) T/F: The above shows how a picture built from two triangular numbers, doubles it._____.

b)  This number (12) is the product of the two consecutive numbers, ___ and ___.

*c) This number (12) is also the sum of the original triangular number ___, and its square _____. 

For the examples on the next page use formulas or pictures below to help find answers.

$S = n^2$ $\qquad$ $S = n^2 + n$ $\qquad$ $S = \dfrac{n^2 + n}{2}$ or $\frac{1}{2} \times (n^2 + n)$



A B C

Diagram **B** has a horizontal line for bisecting the group of small circles while **C** has a vertical line for the same purpose.

13) T/F: The last two diagrams back at exercise 12) suggest that $\mathbf{n \times (n + 1) = n^2 + n}$.___

14) Use the appropriate diagram or formula to find the sum of the first:

a) 16 natural numbers

b) 25 natural numbers

c) 71 natural numbers

d) 71 even numbers

e) 71 odd numbers

f) 120 odd numbers

g) 120 even numbers

h) 120 natural numbers

i) 35 odd numbers

j) 35 natural numbers

*15) Devise a method and find the sum of the natural numbers from 501 to 1000.

MORE SERIES **Unit 16

Answers 1-15:

1) 21, 28, 36, 45

2) 7, 8, 9, 10

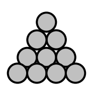
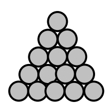
3) 49, 64, 81, 100

4) 56, 72, 90, 110

5) 28, 36, 45, 55

6) $s = \frac{n^2 + n}{2}$ or $\frac{1}{2} \times (n^2 + n)$

7) 36 **8)** 10100 **9)** 45 **10)** T

11) 1, 3, 6 10, 15

12a) T **b)** 3, 4 **c)** 3, 9

13) T

14a) $\frac{1}{2} \times (16^2 + 16) = \frac{1}{2} \times 272 = 136$

b) $\frac{1}{2} \times (25^2 + 25) = 325$

c) $\frac{1}{2} \times (71^2 + 71) = \frac{1}{2} \times (5041 + 71) = 2556$

d) $71^2 + 71 = 5112$

e) $71^2 = 5041$

f) $120^2 = 14{,}400$

g) $120^2 + 120 = 14520$

h) $\frac{1}{2} \times (120^2 + 120) = 7260$

i) $35^2 = 1225$

j) $\frac{1}{2} \times (35^2 + 35) = 630$

***15)** Find the sum of the natural numbers 1 – 1000. From this, subtract the sum 1-500:

$$\frac{1}{2} \times (1000 + 1000^2) - \frac{1}{2} \times (500 + 500^2) =$$
$$\frac{1}{2} \times 1{,}001{,}000 - \frac{1}{2} \times 250{,}500 =$$
$$500{,}500 - 125{,}250 =$$

375,250 (Answer)

SETS, SUBSETS, POWER SETS AND UPSETS

**Unit 17

Subsets

For such a tiny word, “set” is one of the most powerful in the English language. A standard dictionary gives over one hundred meanings for “set”**:** get set go, set the table, a television set, and many more. In mathematics a set is simply a collection of objects - chairs, whole numbers, good ideas, bad ideas, boys less than five feet tall, etc. A set can be enumerated: {a, b, c}; or described: “The set of the first three letters of the alphabet”, and given a simple name such as: A = {a, b, c}, and called set A.

In exercises 1 - 4, enumerate the following sets using the curly brackets:

1) The set of New England states south of Massachusetts.___________________________
2) The odd numbers between 1008 and 1014 ______________________________________
3) The set of prime numbers in the forties (Recall that a prime is a counting number having only itself and 1 as factors, like 7, 13, 29, etc.) ______________________________
4) The set of every even prime. _________________

The set of all multiples of 7 between 30 and 34 is empty. There are no multiples of 7 between 30 and 34. The symbol { } expresses the empty set. You can also use ∅ as a symbol for the empty set. It resembles a zero with a line through it. ∅ = { }. There is only one empty set, also called the “null set”. It can be empty of anything you choose, because it is empty of everything. So of course there is only one such set.

One or more of examples 5 - 8 may not be fair. Wherever you think so, answer NF.

5) B = {♠, 📪, ✸, ✉}. Which two members of set B have a direct connection with each other?

____

6) Give the set of whole numbers larger than 100. ___________________________

7) Give the set of multiples of 13 in the 80’s. ______________________

8) What is the set of all whole numbers which, when squared, give themselves as answers?

____

SETS, SUBSETS, POWER SETS AND UPSETS

**Unit 17

Answers 1 – 8

1) Rhode Island and Connecticut

2) {1009, 1011, 1013}

3) {41, 43, 47}

4) {2}

5) Envelope, mailbox.

6) Perhaps NF, but there is an acceptable way to do it: {101, 102, 103, 104 **. . .** }.

The three dots say "The pattern continues without end".

7) { }, or ∅, or null set, or empty set

8) {0, 1}

Subsets

A = {3, 4, 5} B = {2, 3, 4, 5, 6, 7}

Definition: A is a subset of B if, and only if, every member of A is a member of B. The symbol for this is $A \subseteq B$. It can also be written $\{3, 4, 5\} \subseteq \{2, 3, 4, 5, 6, 7\}$. Important note: $A \subseteq A$.

Surprised? It fits the definition, because every member of A is a member of A. Of course!

So, in mathematics:

Every set has itself as a subset. $A \subseteq A$

Also, about the null set: Every set has ∅ as a subset. $\varnothing \subseteq A$

When asked to list the subsets of a particular set, be sure always to include the set itself and the null set. Here is set M, the set of subsets of the set {a, b}:

$$M = \big\{\{a, b\}, \{a\}, \{b\}, \{\ \}\big\}$$

Notice that set M is a set of sets. It has members which are sets.

SETS, SUBSETS, POWER SETS AND UPSETS **Unit 17

9) List all of the subsets of {a, b, c}. Call it set S. Be sure to get them all. There are 8.

S = {{

The answer to exercise 9 is S = {{a, b, c}, {a, b}, {a, c}, {b, c}, {a}, {b}, {c}, { }}.

The order of the subsets as shown is not required but notice the order, large to small. It suggests a system which will help in the next listing example that you have. Now, guess what. Your next listing example is Exercise 11. Set F = {a, b, c, d}. It has four elements (members) and you are going to list all of its subsets. You saw that a set with 2 elements, {a, b}, has 4 subsets and a set with 3 elements, {a, b, c}, has 8 subsets.

Fill out the following table which will help with Exercise 11.

10)

Number of elements	Number of subsets
0 (null set)	____
1	____
2	4
3	8
4	____

Your answers to exercise 10 should have been one, two, and sixteen. It's easy to forget that $\varnothing$ has itself as a subset, and that a set with one member would have itself and $\varnothing$ for 2 subsets. The answer sixteen is suggested by the pattern of correct answers before it.

11) Now list the 16 subsets of the set F = {a, b, c, d}. Let them be the elements of set Z.
Don't forget sets like {b} which have only one member. Z is called the **power set** of F.

Z = {{____

____

SETS, SUBSETS, POWER SETS AND UPSETS **Unit 17

12) Note that the answers in exercise 10 turn out to be powers of 2. The table in exercise 10 could be extended to give the number of subsets in larger and larger sets. How many subsets are there in a set with**:** (Answer NF if not fair**.**)

a) 5 elements? ____ b) 7 elements? ____ c) 10 elements? _____ d) 24 elements? ______

Answers 11 – 12

11) Z = {{a, b, c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, {a, b}, {a, c}, {a, d}, --- {b, c}, {b, d}, {c, d}, {a}, {b}, {c}, {d}, ∅}.

Did you get them all? It might be worthwhile to inspect the pattern of answers here and decide whether you like your pattern better, if yours was different.

12a) 32 **b)** 128 **c)** 1024 ***d)** NF, but 2^{24} = 16,777,216 is correct.

These exercises were not hard whenever it was easy to extend the table. This is not really fair unless you thought to use the y^x key on your calculator to find 2^{24}.

For 2^{24}, press $\boxed{2}$ $\boxed{y^x}$ $\boxed{24}$ $\boxed{=}$ (16,777,216). (Maybe you had already done it this way.)

or^

SETS, SUBSETS, POWER SETS AND UPSETS **Unit 17

Power Sets

The set of subsets for a given set is called its **power set** and includes the set itself and ∅. That name fits well because the power set grows large very quickly as the given set gets more members. Complete the work in the chart below. It will help to understand that, for a set having "n" members, **2^n gives the number of subsets in its power set.**

13)

Number of elements	Possible Number of subsets	Number of sets in Power Set
0 (null set}	1	$2^0 = 1$
1	2	$2^1 = 2$
2	4	$2^2 =$
3	8	$2^3 =$
4	16	
5	32	
6	64	
7	128	
10	Use calculator →	↓
15	x	↓
22	x	↓
→ → → ___	x	$2^{----} = 33{,}554{,}432$

Make guesses and confirm with calculator

SETS, SUBSETS, POWER SETS AND UPSETS **Unit 17

Answer 13

13)

Number of elements	Possible Number of subsets	Number of sets in Power Set
0 (null set}	1	$2^0 = 1$
1	2	$2^1 = 2$
2	4	$2^2 = 4$
3	8	$2^3 = 8$
4	16	$2^4 = 16$
5	32	$2^5 = 32$
6	64	$2^6 = 64$
7	128	$2^7 = 128$
10	Use calculator →	$2^{10} = 1024$
15	x ↓	$2^{15} = 32768$
22	x	$2^{22} = 4,194,304$
? →→ <u>25</u>	x	$2^{25} = 33,554,432$

Be attentive in what follows. The meanings of the elements and their relationships are not always easy to grasp exactly, but are definite just the same**.**

A = {3, 4, 5}   B = {2, 3, 4, 5, 6, 7}

Let P = {7, 8, 9} and let M = {3, 4, {7, 8, 9}}. **Notice** that <u>set P</u> is a <u>member</u> of set M.   <u>P is not a subset of M</u>. We could also name M as {3, 4, P}, because P = {7, 8, 9}.

<u>A new symbol</u>: **∈.**   P ∈ M. (P is an element, or member, of M)

8 ∈ (is an element of) {7, 8, 9}, but 8 ∉ (is not an element of) M. 8 ∈ P

4 and {4} are different. 4 is a number but {4} is not. {4} is a set whose only member is 4.
So, what about {{4}}? {{4}} is a set whose member is a set whose member is 4. Got it?

14) (T/F) a) 3 ∈ M ___ b) {3} ∈ M ___ c) 4 ∉ M ___ d) 4 ⊆ M ___ e){4}⊆ M___
f) {3, 4}⊆ M ___ *g) {7, 8, 9} ⊆ M ___ *h) {7, 8, 9} ∈ M ___ *i) {{7, 8, 9}}⊆ M ___

15) Let set Q = {{3, 4, 5}}. List the member(s) of set Q. ________________(Be careful)

SETS, SUBSETS, POWER SETS AND UPSETS **Unit 17

Answers 14 – 15

Correct exercises 14 & 15 thoughtfully.

14a) T **b)** F. {3} is a subset of M **c)** F **d)** F **e)** T **f)** T ***g)** F. {7,8,9} $\in$ M

***h)** T *** i)** T Tricky. Similar to part e. Since {7,8,9} $\in$ M, then {{7,8,9}} $\subseteq$ M.

15) The single member of set Q is {3,4,5}, or set A. A $\in$ Q.

A = {3, 4, 5} B = {2, 3, 4, 5, 6, 7} C = {3, 4, {4, 5}} D = {all proper subsets of B}

Some Important Ideas Repeated: . 5 $\in$ {4 ,5} but 5 $\notin$ (is not a member) of C.

Set C has three members: 3, 4 and the set {4, 5}. Nothing prevents a set from being a member of another set. {4,5} $\in$ C but {4,5} $\subseteq$ B.

If Q = {6, {5, 8}} then 6 $\in$ Q but {6} $\subseteq$ Q.

W = The set of whole numbers (0, 1, 2, 3, . . .}.

N = The set of natural, or counting numbers (1, 2, 3, 4, . . .}.

I = The set of integers {. . . -3, -2, -1, 0, +1,+2, +3 . . .}

Another new symbol: $\subset$. A$\subset$B is often read "A is a proper subset of B". It is a little different from A $\subseteq$ B. For a set to be a proper subset ($\subset$) of another, it must be missing at least one element of the other. {3, 5, 7} $\subseteq$ {3, 5, 7} is okay but {3, 5, 7}$\subset$ {3, 5, 7} is not okay. {3, 5}$\subset$ {3, 5, 7} is okay, and of course, so is {3, 5} $\subseteq$ {3, 5, 7}.

Notes: { } is not the same as {{ }}. { } is the empty set but {{ }} is not empty.
{{ }} has a member which is { } or $\varnothing$. {{ }} $\neq \varnothing$. {{ }} = {$\varnothing$}. $\varnothing \in$ {{ }}.

Z $\subseteq$ Z is true, but Z $\subset$ Z is false no matter what set Z represents unless Z is said to represent the null- set. Do you see that?

S
T
U
D
Y

Exercises 16 - 26 are True or False: (Tricky)

16) A $\subset$ I___ 17) C $\subset$ I ___ 18) {4, 5} $\subset$ C ___ 19) {4, 5} $\in$ C___

20) {4, 5} $\in$ I ___ 21) {4, 5}$\subset$ I ___ 22) 4 $\subset$ C___ 23) 3 $\notin$ C ___

24) C $\subseteq$ B ___ 25) C $\subset$ C ___ 26) C $\subseteq$ C ___

27) How many members has set D? ______

SETS, SUBSETS, POWER SETS AND UPSETS **Unit 17

28a) How many members has the power set of set A? ______

*b) How many members has the power set of the power set of set A? ______

29) 10^{123} is an element of which of the sets A-D, W,N,I ?____________________

30) The English alphabet has 26 letters.

*a) How many different subsets of exactly 25 different letters each can be made? (Note that exactly 1 letter will be missing from the alphabet at a time)______

b) Use your calculator to determine the number of sets in the power set of the alphabet. _____

*31) The number of sets in the power set of a set with 10^{123} elements is $2^{\square}$.

Answers 16 – 31

Correct these thoughtfully. Expect mistakes.

16) T **17)** F. The set $\{4,5\} \notin$ I. **18)** F. $\{4,5\} \in$ C. **19)** T **20)** F

21) T **22)** F. $\{4\} \subseteq$ C but $4 \in$ C **23)** F **24)** F, because $\{4,5\} \notin$ B

25) F **26)** T **27)** 63 (The power set has $2^6 = 64$ members.)

28a) $2^3 = 8$ **b)** 256 ($2^3 = 8$, $2^8 = 256$)

29) 10^{123} is in all of sets I, N, and W if you extend them far enough, as the dots indicate that you should (at least in your imagination).

30a) 26 **b)** $2^{26} = 67{,}108{,}864$

31) The number in the box should be 10^{123} although it won't fit easily. 10^{123} becomes the exponent of 2**:** $2^{(10^{123})}$

ACHILLES AND THE TORTOISE *Unit 18

Over 2000 years ago there lived a wise philosopher named Zeno who liked to confuse people by making up little mathematical stories which were very difficult to explain. Zeno was so clever at this that he mystified and embarrassed mathematicians for centuries because they could not explain his paradoxes. A paradox is a situation which seems fine, but on further investigation, seems impossible. The most famous of his paradoxes is the story of “Achilles and the Tortoise”. In this story, Achilles runs a race with a tortoise. In order to appreciate what a ridiculous situation this is, you must realize that Achilles was a fleet-footed warrior of Greek myth and the tortoise, of course, was nothing more than a turtle. He was not a very smart turtle either or he would never have tried to race the great Achilles. It may be that the tortoise knew something that Achilles did not.

The day of the race arrived and Achilles, being a generous fellow, allowed the poor tortoise a head start of 100 meters. For this, the tortoise was very grateful because he thought that surely Achilles would never catch him before he crossed the finish line if he had such a head start.

The race began with both getting off to a good start at the same time. It soon became clear that Achilles could run about ten times as fast as the tortoise. With a laugh, Achilles bounded across the field to the point where the tortoise had started. This distance, of course, was 100 meters, the amount of the head start. Meanwhile, the tortoise, being able to run only 1/10 as fast as Achilles, plodded along a distance of 10 meters. (We will assume for the sake of simplicity that our measures exactly reflect the actual distances; untrue of course). Achilles thought to himself that now the race would be over in a matter of seconds as he quickly ran the ten meters which the tortoise had covered.

While Achilles was covering this 10 meters, the tortoise, still traveling at 1/10 the speed of Achilles, has had time to amble 1 meter. Achilles covered this one meter. During this time, the very stubborn tortoise was able to travel 0.1 meter (1/10 meter). Achilles, perhaps getting frustrated at the situation by now, traveled this 0.1 meter and so the tortoise had time to travel 0.01 meter.

Poor Achilles! When is all this going to end? Every time he covers the distance which the tortoise has covered, the tortoise uses this time to cover a little more distance. In other words, no matter how hard this great Greek warrior tries to close the gap between himself and the tortoise,

ACHILLES AND THE TORTOISE *Unit 18

the tortoise seems to be able to travel 1/10 of this distance and prevent Achilles from overtaking him.

Let's pause a minute and think about this situation. At the beginning of the story it seemed obvious that Achilles would have no difficulty in overtaking the tortoise. Perhaps now you are not so sure. Think about each of the following questions carefully before you answer it.

1) Do you think that it is necessary for Achilles to put on an additional burst of speed in order to catch the tortoise? _________
2) If, all of a sudden, Achilles begins to run twice as fast as he has been running, do you think that the old problem still remains, even though with different numbers? That is, as Achilles closes the distance between himself and the tortoise, will the tortoise still be able to cover a little ground while Achilles does this? (Yes/no) ___________
3) If the situation continues as it was when we left it, do you think that Achilles will ever catch the tortoise? __________
4) Now try to forget about all this hocus-pocus about 1/10 as fast and so forth. What does your common sense tell you about Achilles? (Circle one)
   a) He catches the tortoise very quickly.
   b) He catches the tortoise, but only after a long, hard chase.
   c) Poor Achilles never catches the tortoise.

Before we go any further, let's get something straight. If Achilles can't catch the tortoise, then a jet plane can't catch a snail. If Achilles can't catch the tortoise, then there is no such thing as a race; nothing can catch anything, fast or slow. Achilles, of course, does catch the tortoise with no difficulty at all. Our everyday experience tells us that this must be true. (You might want to go back now and change an answer or two.) The fact remains, however, that we have not explained the paradox. We have not even discovered how far Achilles travels before he actually catches the tortoise. Let's back both runners up to their starting lines and watch Achilles try to overtake the tortoise again.

ACHILLES AND THE TORTOISE *Unit 18

This time, however, we will keep track of the distances that Achilles covers.

a)	Achilles runs to the tortoise's starting point (Meanwhile, the tortoise runs 10 meters)	100 meters
b)	Achilles covers these 10 meters. (Meanwhile, the tortoise runs 1 meter)	10 meters
c)	Achilles covers this 1 meter (Tortoise: .1meter)	1 meter
d)	Achilles: .1 meter (Tortoise: .01 meter)	.1 meter
e)	Achilles: .01 meter . (Tortoise: .001 meter)	.01 meter
	etc.	.001 meter
	etc.	etc.
	etc.	etc.

5) Do you begin to see a pattern in the distances which Achilles is covering? What is the next distance after .001?_________ and the next? ____________. We could go on for as long as we please, and the further we go, the smaller the distance.

6) Write the <u>next</u> <u>five</u> Achilles terms and then add *all* those Achilles terms shown. Finally, indicate that the sum is non-ending.

7)

```
   100.
    10.
     1.
      .1
      .01
      .001
      .__________
      .__________
      .___________
      .___________
+     .
____________________
```

The addition exercises to the left shows stages of Achilles race. But, during Achilles' first stage where he travels 100 meters, the tortoise travels 7) _____ meters, the second number in Achilles' stage. In fact, we can see that pattern continue to the bottom, with stages like those seen in exercises 4 and 5 above. And although we choose to end the addition with 5 more stages, we will trace it much further on the next page without being hampered by a limit of space. Imagination is all that we will need to make the problem clearer, much larger, and much more mathematical than arithmetical.

ACHILLES AND THE TORTOISE *Unit 18

Back to our addition. Your answer should have been 111.11111111 . . . (Yes, you need the three little dots.) Let's think about this rather surprising result.

The dots again indicate that this number continues for an infinite number of decimal places. The strange thing about such a number is that no matter how many ones we tack onto the end of it, **the number never reaches 112**. As a matter of fact, it never reaches 111.2.

Important!

*8) Write the smallest two-place decimal which our peculiar number never reaches.___________

Well, just how big is this number? *Exactly* how far does Achilles travel before catching the tortoise? You didn't forget about Achilles, did you? Our strange number is getting larger and larger but we know it can never get as large as 111.2 no matter how many ones we write. In other words, there is a limit to how big 111.1111111111111111111111111. . . can become merely by writing more ones. The question is, exactly what is that limit?

Here are some exercises which will help you find the *exact limit*.

9) Change 1/9 to a decimal. Carry out to 3 decimal places. $9\overline{)1.000}$

(Recall that 1/9 means $9\overline{)1\quad\quad}$)

10) Now express 1/9 correct to ten decimal places without bothering with division:___________

11) What fraction is the limit of .1111111111. . . ? That is, if you continue tacking many ones onto it, to what fraction does it draw nearer and nearer ? ___________

Hint: Peek back at exercise 9

12) What is the limit of 111.1111111111 . . . ? _______________
(This answer tells you how far Achilles traveled to overtake the tortoise. Exactly.)

*13) Here's a tricky one. Exactly how far had the tortoise traveled when Achilles overtook it?_____________

ACHILLES AND THE TORTOISE *Unit 18

14) On the line below, represent Achilles' first stage of the race. This is 100 meters and can be shown conveniently by a length of 100 millimeters (10 cm.). Use a metric ruler and put a neat little vertical mark 100 mm to the right from the start.

Start

15) On the same line, carefully represent Achilles' second and third stages of the race.

It might be surprising to see Achilles' stages choke off so quickly. If you were asked to represent the fourth stage you would have to put that mark right up against your stage three mark. Any marks for further stages probably would only make a mess.

Because Achilles actually passes the tortoise after traveling $111\frac{1}{9}$ meters, we will dispense with the word limit and simply say that 111.1111 . . . = $111\frac{1}{9}$. Achilles does achieve that distance and exceed it; he doesn't merely get close to it.

16) Since 1/9 = .1111 . . . , what is the infinitely repeating decimal for 2/9?__________

17) Working mentally, give answer as an infinitely repeating decimal, or whole number and repeating decimal:

a) $\frac{7}{9}$ = _________ b) $1\frac{5}{9}$ = ___________ c) $\frac{17}{9}$ = ______________

18a) .444 . . . = ____(What fraction?) b) .333 . . . = _____(Reduce this fraction.)

c) .666 . . . = ____ d) .999 . . . = _____ Simplify.

(Do you really believe this?)

ACHILLES AND THE TORTOISE *Unit 18

Answers

1) no (The correct opinion) **2)** yes **3)** yes **4)** a

5) .0001, .00001

6) 111.11111111 . . . (and you really should have the dots . . .) **7)** 10

8) 111.12 **9)** .111. **10)** .1111111111 (Note: Exercise 10's answer is rounded)

11) 1/9 **12)** 111 1/9 **13)** 11 1/9 meters

14) and **15)**

16) .222 . . . (Any number of 2's is okay.)

17a) .777 . . . **b)** 1.555 . . . **c)** 1.888 . . .

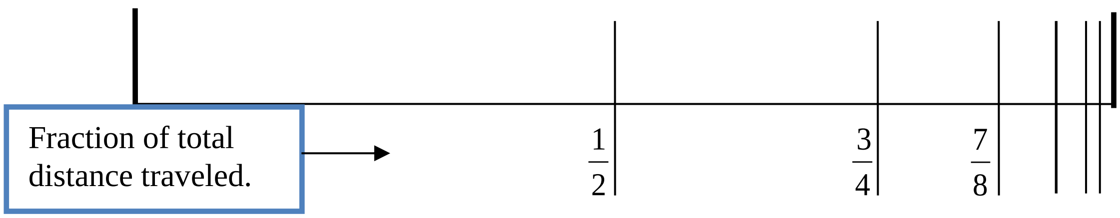
18a) 4/9 **b)** 1/3 **c)** 2/3 **d)** 9/9 or 1. (Yes, it's true; .999 . . . does equal 1, not just almost. You might be tempted to think of it as "not quite". Not so. More later for skeptics – see Unit 33.)

Note that the decrease in the distance of each stage means that the *time* of each stage also shortens and approaches 0. More about this in the next unit, *Halfwalk.*

HALFWALK **Unit 19

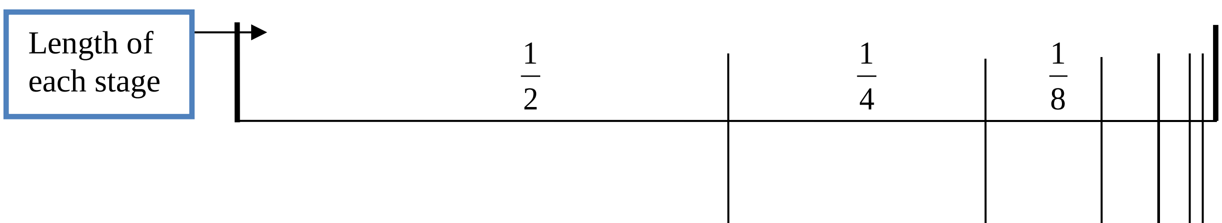
Here is a problem about the infinite which you can experience by walking across the room. It is also a journey through patterns of the infinite.

Stand with your back against the wall and walk halfway (approximately) across the room. Stop for a brief moment and then walk half of the remaining distance, pausing again. Walk half of the new remaining distance and keep doing this for the rest of your life or until you reach the opposite wall, whichever comes first.



In theory, you might think the walk would take forever, but in a practical sense, physically, it shouldn't take much time.

1) Above, on the diagram of your walk, write the next three fractions in the sequence following $\frac{7}{8}$. You won't have room for the last one, so put it below or to the right of its mark. **Answers** at the top of page 2



2) On the second figure, the length of each of the first 3 stages is recorded. Of course they get smaller and smaller. On the diagram, write the next three fractions in this sequence, and realize that the final space is reserved for further stages which get too tiny to show.

3) The larger and larger numbers we see in the <u>top</u> diagram are the <u>accumulations</u> of the distances shown in the bottom diagram**:** $\frac{1}{2}+\frac{1}{4}=\frac{3}{4},\quad \frac{3}{4}+\frac{1}{8}=\frac{7}{8},\quad \frac{7}{8}+\frac{1}{16}=\frac{15}{16},\ .\ .\ .$

Write the next three additions with answers in the same form as immediately above.

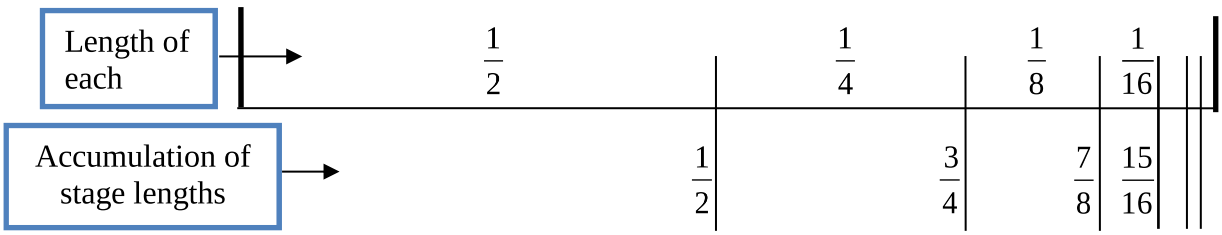
______________________ ____________________ _____________________

HALFWALK **Unit 19

Answers 1 – 3

1) $\frac{15}{16}, \frac{31}{32}, \frac{63}{64}$ **2)** $\frac{1}{16}, \frac{1}{32}, \frac{1}{64}$ **3)** $\frac{15}{16} + \frac{1}{32} = \frac{31}{32}, \frac{31}{32} + \frac{1}{64} = \frac{63}{64}, \frac{63}{64} + \frac{1}{128} = \frac{127}{128}$

These numbers below should have a familiar look to them. The denominators are all powers of 2. There are clear patterns to follow in extending the stages.



The diagram above combines the two diagrams from page 1 and carries their numbers one stage further. In example 3, you carried those same numbers 2 stages further than that. So, we have recorded the first few stages of your walk across the room with lengths getting ever smaller. This is much like Achilles and the Tortoise, but you are chasing a stationary wall rather than a moving tortoise. Also, you paused at each "stage", but Achilles did not. We should admit at this point that the mathematics we are discussing is exact and therefore does not reflect what actually happened when you crossed the room. Probably you did not actually pause at $\frac{127}{128}$*ths* of the walk.

<u>Let's eliminate the pauses</u>. They were put in to make the stages easier to visual. The stages and their mathematics remain. Even if you tried, you could not be sure of your *exact* position at any time because of the <u>impossibility of exact measure.</u>

4a) How far is it to the wall (in theory) after $\frac{127}{128}$*ths* of the room-width walk? ________

b) How many stages does it take to go $\frac{127}{128}$*ths* of the room width? ______(Most of them are on the diagram.)

c) $128 = 2^{\square}$ d) Are your answers to b) and c) the same? ______ They should be.

HALFWALK **Unit 19

We know the length of each stage (above the line) and we have accumulated their sum as we went along (below the line). You carried the accumulated sum to $\frac{127}{128}$.

5a) $\frac{127}{128} = \frac{127}{2^{\square}}$ b) What is the accumulated sum at the end of the next stage?______

The answer to exercise 5a) is ^{7}, that is, $\frac{127}{2^7}$. The answer to 5b is $\frac{255}{2^8}$, or $\frac{255}{256}$.

We often express such numbers as $\frac{2^7 - 1}{2^7}$ and $\frac{2^8 - 1}{2^8}$.

Note that the numerator is 1 smaller than the larger denominator. This gives a number <u>very</u> close to 1.

<u>Using this form</u>, give the accumulated sum of:

6a) 10 stages of the walk. ______ b) 60 stages.________

*7) Use an exponent and fraction and tell how close to the wall you are (in theory) after:

a) 10 stages ________ b) 60 stages ________ c) 200 stages ___________

*8) Using exponents, tell the accumulated sum of 200 stages: ____________ Using exponents to express very small numbers like those in exercise7b and 7c is convenient after getting used to it, but, we tend to lose a sense of how large or small the numbers really are. For instance, the 60 stages in ex. 7b leave you $\frac{1}{2^{60}}$*th* of the room width away from the wall. That number is about $\frac{1}{1152900000000000000}$*th* of the room width from the wall after 60 stages. That's about <u>one quintillionth</u> of the room width.

9) Use calculator: How close to the wall are you after 26 stages? Use $\boxed{y^x}$ key and give answer with a fraction having no exponent: ________________

10) And what fractional part of the room has been covered in 26 stages? No exponents or decimals in answer: ______________________________

No matter what **finite** number of stages are added and expressed as in exercise 10, $\frac{67108863}{67108864}$ (Did you get this for exercise 10?), the sum still falls short of the wall. We are getting closer and closer to 1 whole room width.

No matter how tiny the distance to the wall, in this case, 1/67108864*th* room width (exercise 9 answer), someone can always think of a smaller one, for instance 1/6108865. This number is getting closer and closer to 0. In this kind of definite situation we know that there exists a particular number to which we are getting closer and closer but cannot pass. The mathematician calls this number a **limit**. This is a crude definition of a limit but is all we need for now.

11) The accumulated sums are getting larger and larger, but by a *smaller amount* each time. What is the limit being **approached** by the accumulated sums? _____ room width(s). Although the individual distances get smaller and smaller, the sum of these is getting ever larger, but not necessarily without limit**:**

Limit of the sum $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}$ . . . = 1**.** So is 1 the answer to exercise 11? We will (probably) accept 1 as the answer so long as we remember our notion of limit is limited!

We have seen that seven stages of the trip across the room amount to $\frac{2^7-1}{2^7}$ *ths* of the trip, leaving $\frac{1}{2^7}$ remaining (See exercise 5 and 6). The number of the stage (^{7}) appears as an exponent in calculating the distance traveled and remaining. The *decimal* part of the total trip across the room can be figured easily on a calculator once you have the fractional part expressed, $\frac{255}{256}$. (Exercise 6). Do it. ___________. In case you have forgotten, the division symbol ÷ says visually a number over a number means to divide them, top by bottom. Your answer should have been 0.9960938, or 0.99609375 with a ten digit calculator.

Answers 4 – 11

4a) $\frac{1}{128}$ **b)** 7 **c)** 7 **d)** yes

5a) 7 **b)** $\frac{255}{256}$

6a) $\frac{2^{10}-1}{2^{10}}$ **b)** $\frac{2^{60}-1}{2^{60}}$

7a) $\frac{1}{2^{10}}$ **b)** $\frac{1}{2^{60}}$ **c)** $\frac{1}{2^{200}}$

8) $\frac{2^{200}-1}{2^{200}}$

9-11) Answers are in the discussion.

For the next exercise <u>complete all blanks in the chart on the next page.</u> It will take some time to work out the decimal values for some of the stages. You will need your calculator and lots of patience for division and for the $\boxed{x^n}$ or ^ keys. It is best to do <u>all</u> of your work in rows from left to right. Ideas for shortening calculator work might occur to you. As a column of decimal answers develops stage by stage, pay some attention to how the decimals are changing and whether results seem to be making sense. Check answers from time to time before finishing. It is easy to let calculator work complicate your efforts to make sense. You will probably get more efficient as you go along. Discussion of results are on page 7 of this Unit. Use the decimal column that suits *your* calculator, 8 or 10 places. Leave out the other one. In all members of the "Fractions" column the top and bottom differ by one. Try using this to make shortcuts in your work, not for rushing, but for more efficient computation.

HALFWALK **Unit 19

Answers will vary slightly with different calculators.

**12)

Stage	Formula	Fraction	8 place Decimal	10 place Decimal
16	$\frac{2^{16}-1}{2^{16}}$	$\frac{65535}{65536}$	0.9999847	0.999984741
17			0.9999924	0.999992371
24	$\frac{2^{24}-1}{2^{24}}$			
25	$\frac{2^{25}-1}{2^{25}}$		(Rounded to 1.0 by calculator)	
26	$\frac{2^{26}-1}{2^{26}}$	$\frac{67108863}{67108864}$		
29	$\frac{2^{29}-1}{2^{29}}$			0.9999999998
30				
31				

Note Stages

Do all

We have already mentioned that an important difference between the “Halfwalk” situation and that of Achilles and the Tortoise seems to be that the tortoise moves while the wall does not. Since it seems almost impossible to believe that a swift athlete cannot overtake a plodding tortoise, it might be easier to believe that a person could not "catch" a stationary wall. If we could move in perfect accordance with the mathematics, *maybe* we would never reach the wall. In other words, *theoretically* we might never reach the wall.

Answers 12:

Answers will vary slightly because of calculator differences.

For ease in correcting, # indicates your answers.

****12)**

Note Stages

Stage	Formula	Fraction	8 place Decimal	10 place Decimal
16	$\frac{2^{16}-1}{2^{16}}$	$\frac{65535}{65536}$	0.9999847	0.999984741
17	# $\frac{2^{17}-1}{2^{17}}$	# $\frac{131071}{131072}$	0.9999924	0.999992371
24	$\frac{2^{24}-1}{2^{24}}$	# $\frac{16777215}{16777216}$	# 0.9999999	# 0.999999994
25	$\frac{2^{25}-1}{2^{25}}$	# $\frac{33554431}{33554432}$	1.0 (Rounded to 1.0 by calculator)	#0.999999997
26	$\frac{2^{26}-1}{2^{26}}$	$\frac{67108863}{67108864}$		#0.999999985
29	$\frac{2^{29}-1}{2^{29}}$	# $\frac{536870911}{536870912}$		0.9999999998
30	# $\frac{2^{30}-1}{2^{30}}$	# $\frac{1073741823}{1073741824}$		#1.0
31	# $\frac{2^{31}-1}{2^{31}}$	# $\frac{2147483647}{2147483648}$		#1.0

HALFWALK **Unit 19

So how did you do? Were you able to form all the fractions needed? And did you work out a reasonably efficient calculator procedure to produce the decimal equivalents for the fractions? Resist the temptation to shrug off errors. Come back to them later, if needed.

Albert Einstein, the great physicist of the 20th century, loved to conduct for himself what he called mental experiments. He would picture himself, for instance, in outer space inside a closed elevator with an outsider firing rifle shots at the elevator. Einstein would observe the path of an imaginary bullet as it passed though the elevator and come to conclusions about the movements of the elevator in relation to the location and movements of the person firing the rifle. We can do that too, but we do not need a rifle. We need a space log to sit on and a pair of super-eagle X-ray binoculars to enable us to have sharp focus on a small area, like a bird of prey watching a tiny field mouse from a great height.

So here we are, suspended in space, watching the earth turn beneath us from our left to our right. We have X-ray ability to see what people are doing under their roofs. We suddenly see a person chasing a wall. The wall is moving from our left to our right with the earth and so is the person "chasing" it. Both are moving, like Achilles and the tortoise, so we know the wall will soon be overtaken by the person. You probably are witnessing that event in your “mind’s eye” and might feel you no longer need to entertain serious doubts about reaching the wall (if you ever did have doubts).

Remember from Achilles that the *times* required to grow those tinier and tinier distances *themselves* grow tinier and tinier right along *with* the distances. We do not slow down as we approach the overtake point in *Achilles* or the wall in *Halfwalk*. *Achilles* and *Halfwalk* are fundamentally the same. Both were considered in stages to allow us a model for studying changing sequences. These stages are mathematically okay but they introduce deception into our visualizing of the walk or race. They also might cause us to wonder about the nature of infinity in mathematics.

RECOLLECTIONS AND EXTENSIONS II

Unit 20

1) In the unit "Halfwalk" let's suppose that the room was 16 feet wide. The series of halfwalk stages across the room measured in feet would then begin like this:

$$8 + 4 + \ . \ . \ . \ .$$

a) Continue the above series for 8 more terms. Use fractions without exponents, like $\frac{1}{16}$, when you get below 1. Your last term above should be $\frac{1}{64}$.

b) What is the sum of the series you wrote, including 8 + 4 +. . .?. ______. Notice the pattern after writing some more of the series. It will ease the adding.

c) What is the limit of the sum of this series as the denominator gets larger without limit? _____

*2) The sum $4 + 1 + \frac{1}{2} + \frac{1}{8}$ is written in the binary system this way: 101.101_2

a) Do the same for the 10 term series in 1a). ___________________

*b) Change 14.75 to the binary system. _____________________

3) Write the reciprocals of: (Reduce any fraction answers.)

a) 1 ½ _____ b) 0.8 _____ c) 8 thousandths ______ d) 5 ten-thousandths _________

4) $1,398,000 \times 10^{-5}$ = __________.

5) In the game of "Bing", how many people should have their hands up immediately after

a) The 7th bing? _______ b) The 8th bing? _______

6. Where does a -6 jumper land after making 16 jumps from 88?_____

7a) $83.9 \times 10^{------} = .00839$ b) $83.9 \div 10^{------} = .00839$

RECOLLECTIONS AND EXTENSIONS II Unit 20

8a) List only the subsets having <u>exactly 2 members each</u> from the set A = {5, 6, 7, 8, 9}. Try to use a system to keep track. There are ten subsets.

b) How many sets in the Power Set of Set A? ____ (A itself is a subset & so is { }.)

9a) 16 - (-16) = ____ b) $16 + 2^4 - 2^3 + 2^5 =$ ___ *c) -16 - $\square$ = 32

10) T / F: a) $\{a\} \subseteq \{a, b, c\}$. ___ b) $\{a\} \in \{a, b, c\}$ ___ c) $a \subseteq \{a, b, c\}$ ___

11) (Definition) A set is <u>infinite</u> if and only if it is in____________________ (See Unit 12)

____

12) How many terms in the set {1, 2, 3, 4, . . . }? Choose: a) $\aleph$ b) Ψ c) $\aleph_0$ ____(See Unit 13)

Very small and very large numbers that one meets in the world of science are written in what is called **scientific notation**, not a surprising name. Astronomy needs large numbers and biological research and atomic science need small numbers. Here are a few.

13) Multiply <u>mentally</u> and write answer as a number between 1 and 10, times $10^?$. Hint: First determine the leading digits and place decimal point if needed.

Samples: $3,000 \times 200,000,000 = 6 \times 10^{11}$ $.000000012 \times .00000012 = 1.44 \times 10^{-15}$

$3,000 \times 400,000,000 = 12 \times 10^{11} = 1.2 \times 10^{12}$. Remember, between 1 and 10, times $10^?$.

a) 3,000 x 3,000,000,000 = ___________ b) 3,000 x 40,000,000 = __________

c) 11,000,000 x 50,000,000 = __________ *d) 12,000,000 x 50,000,000 =__________

e) .000002 x .00000003 = _________ f) .00000011 x .000013 = __________

g) $.123456 \times 10^3 =$ __________________

Now go back and check a) - g) with your calculator. Answers are also given on the <u>next page</u> but don't use them unless you are unsuccessful with your calculator check. Calculators can be tricky. Be persistent and patient. See your calculator manual under "scientific notation" if you need to (and if you still have the calculator manual).

RECOLLECTIONS AND EXTENSIONS II Unit 20

Answers

1a) $(8) + (4) + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$

b) $15\frac{63}{64}$. Did you notice the answer patterns in adding successive terms?

Extensions: Answers on next page

b_1 What would be the answer if you added 1 more term to b), just above?________

b_2 Add three more terms to your b_1 answer. Give the three results:_____, _____,_____,

c) 16

2a) 1111.111111 **b)** 1110.11_2 ($.11_2$ provides ½ + ¼ to give the needed ¾)

3a) 2/3 **b)** 5/4 or 1 ¼ **c)** 125 **d)** 2000

4) 13.98

5a) 3 **b)** 1

6) -8

7a) $^{-4}$ **b)** ^{4}

8a) {5,6}, {5,7}, {5,8}, {5,9}, {6,7}, {6,8}, {6,9}, {7,8}, {7,9}, {8,9} **b)** $2^5 = 32$

9a) 32 **b)** 56 **c)** -48

10a) T **b)** F **c)** F

11) One to one correspondence with part of itself (or with a proper subset of itself).

12) c) $\aleph_0$ Aleph null is Cantor's first transfinite number, remember?

13) Multiply mentally and write answer as a number between 1 and 10 times 10^n (Scientific notation)

Samples: 3,000 x 200,000,000 = 6 x 10^{11} .000000012 x .00000012 = 1.44 x 10^{-15}

3,000 x 400,000,000 = 12 x 10^{11} = **1.2 x 10^{12}**

a) 3,000 x 3,000,000,000 = **9x 10^{12}**

b) 3,000 x 40,000,000 **1.2 x 10^{11}**

c) 11,000,000 x 50,000,000 = **5.5 x 10^{14}**

d) 12,000,000 x 50,000,000 = **6 x 10^{14}**

e) .000002 x .00000003 = **6 x 10^{-14}**

***f)** .00000011 x .000013 = **1.43 x 10^{-12}**

g) .123456 x 10^3 = **123.456 = 1.23456 x 10^2**

RECOLLECTIONS AND EXTENSIONS II Unit 20

Answers to the two questions b_1 and b_2, presented in the answers to exercise 1.

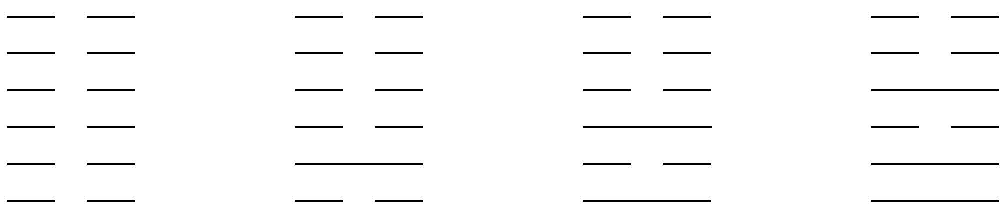
b_1) $15\frac{127}{128}$ ***b_2)** $15\frac{255}{256}$, $15\frac{511}{512}$, $15\frac{1023}{1024}$

No Quiz for Unit 20.

FO-HI AND HIS JE-KIM

Unit 21

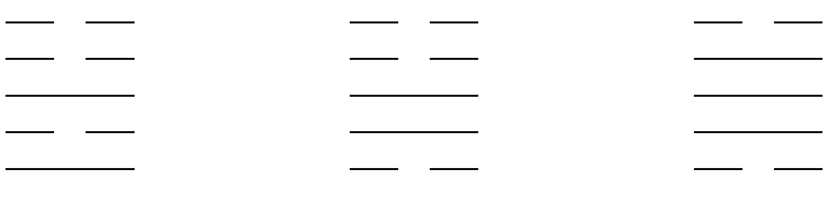
If you think the binary system is new then you have not heard of Fo-Hi. Fo-Hi was a Chinese gentleman who, many centuries ago, wrote a little book called Je-Kim. In it is a system of numeration which looks like the figures below. Of course, the digits 0, 2, 5 and 11 did not appear in his ancient Chinese book.



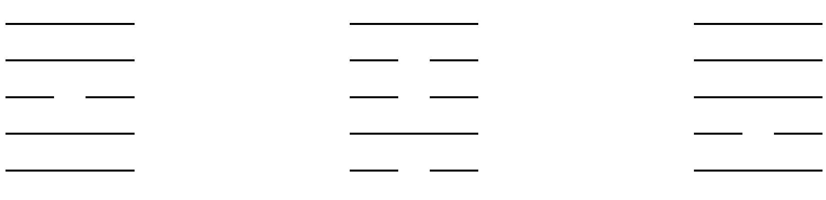
0 2 5 11

Fo-Hi did not explain how his system worked. It took a famous mathematician in Germany named Leibnitz (pronounced "Libe-nits") to figure out what Fo-Hi meant. See, if you can, what Leibnitz discovered. It is not difficult but it does help if you understand the binary system of numeration.

After you crack the code, here are some of Fo-Hi's numerals to change to our decimal system of numeration. Write the Hindu-Arabic numeral underneath.



a) ___ b) ___ c) ___



d) ___ e) ___ f) ___

FO-HI AND HIS JE-KIM — Unit 21

Here are some Hindu-Arabic numerals to convert to Fo-Hi's system. The dotted lines are for help in drawing in darker lines.

g) 9 =

h) 23 =

i) 43 =

j) 10 =

k) 47 =

l) 62 =

m) What is the largest number that can be expressed with this size stack of lines? ____ Do it. ___.

n) 1 + the answer to exercise m) is ___, which equals 2^{----}.

*o) How many lines high must a stack be to designate the number 63 ___?

64 ___? 127? ___ ? 128 ___? 2^{10} ___? 2^{10} - 1___; 2^{10-1} ___? 2^{20} ___?

Answers

a) 10

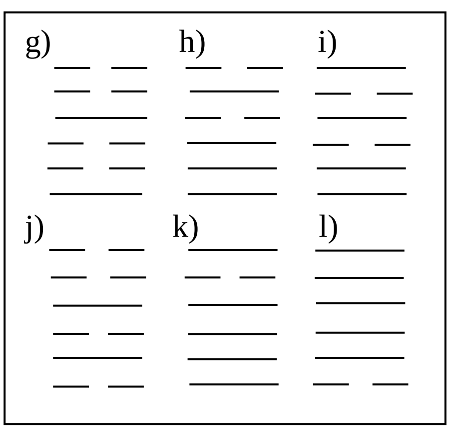
b) 13

c) 28

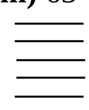
d) 55

e) 37

f) 59



m) 63



n) 64; ^{6}

o) 6; 7; 7; 8; 11; 10; 10; 21.

No Quiz for Unit 21.

CARL FRIEDRICH GAUSS TEACHES THE TEACHER **Unit 22

(Gauss rhymes with house)

There is an interesting childhood story about one of the greatest of all mathematicians. Most accounts have the 9 year old Carl Friedrich Gauss (Friedrich is pronounced Freed-rish) sitting in a one-room school house in the town of Braunschweig, Germany, in the year 1786, just after the end of our Revolutionary War. The schoolmaster, Herr Buttner (pronounced Bewtner), wanted to give his class a task that would keep them occupied during the warm afternoon while he sat at his desk relaxing, perhaps even nodding off for a brief nap.

Herr Buttner came up with what he thought was a perfect plan.

“Take out your slates”, he said.

Paper was not generally available to students in those days. Each student had a small slate and chalk for writing and doing calculations.

“Now", said Herr Buttner, “Add up all the whole numbers from 1 to 200.”

The class was not happy about this as they came to realize the size of the task set for them but soon started to work with no complaints loud enough for Herr Buttner to hear.

The school master settled comfortably behind his desk, watching the students. They would write the first few numbers on the slate, add them, scrub out the addition with a sleeve or rag, and carry the sum to the top of the slate to repeat the process with the next few numbers. Things were going well for Herr Buttner and he leaned back in his chair, well prepared for a nap. Just as he was slipping into slumber a rather loud clatter on his desk jarred the schoolmaster from his barely begun nap. In front of him stood the quiet Carl Friedrich Gauss and on the desk was a single slate.

Herr Buttner came fully awake, glared at Carl Friedrich, and finally motioned him to his seat. While Carl was returning to his seat Herr Buttner had time enough to peek at Carl Friedrich’s work. Written on the slate was the single number 20,100 -- the correct answer! Carl Friedrich had computed in just a few minutes a problem which was supposed to take long enough to permit a sleepy schoolmaster time for a nap. But there was to be no nap that day.

How had Carl Friedrich done the problem so quickly? This was a feat which irritated his teacher but also gained his respect and his wonder. Herr Buttner later sent for more advanced material for Carl Friedrich than any he had at hand. Perhaps this was the beginning of a career in mathematics which eventually ranked him with such renowned great ones as Archimedes and Sir

Isaac Newton. And we must think that the classmates of Carl Friedrich, some older than he was, were curious about his "trick", if that is what it was.

We should understand that probably neither the teacher nor the father of Carl Friedrich Gauss had encouraged creative thinking. Learning in schools of 200 years ago was largely by rote memory and repetitious practice of skills. Of course some of that is needed today, but not at the expense of understanding and a certain desire to know.

What follows reflects some probable ideas of Carl Friedrich Gauss as he worked on the problem. Perhaps you have ideas of your own and would like to try matching wits with the great Gauss before studying his solution below.

Using three dots to represent material left out, the sum is written thus:

$1 + 2 + 3 + \ldots + 198 + 199 + 200$

Right away we can see two things: 1) It had occurred to Carl Friedrich that there must be a better way than adding all those numbers to get their sum. This is key to the astounding original thinking of a nine year old boy who was trained to do what he was told and was expected to follow certain procedures. He must have had a freedom of mind which allowed a path to sure knowledge which would not be denied by expectations of others. 2) Carl Friedrich saw that one can deal with a large number of things by a simple indication of their presence, a symbol (three dots in this case) which says, "We all know what goes here; let's not waste time and clutter things up by writing it all out."

The next thing that Carl Friedrich did was to write the series backwards underneath the shortened version of the series itself.

$$\begin{array}{rrrcrrr} 1\ + & 2\ + & 3\ + & \ldots & +\ 198\ + & 199\ + & 200 \\ 200\ + & 199\ + & 198\ + & \ldots & +\ 3\ + & 2\ + & 1 \end{array}$$

This would only be done by someone with a very keen eye for pattern, symmetry, and balance, probably not instilled by any formal instruction. For Gauss, the problem is essentially solved at this point, even though some rather thoughtful calculations remain.

$$\begin{array}{rrrcrrr} 1\ + & 2\ + & 3\ + & \ldots & +198\ + & 199\ + & 200 \\ 200\ + & 199\ + & 198\ + & \ldots & +\ 3\ + & 2\ + & 1 \\ \hline 201\ + & 201\ + & 201\ + & \ldots & +201\ + & 201\ + & 201 \end{array}$$

CARL FRIEDRICH GAUSS TEACHES THE TEACHER **Unit 22

Keeping in mind that we now have *two* series we notice that each vertical sum is the same, namely 201. Two very long sums have been transformed into a single multiplication because there are 200 of these 201's, even though they are not all written. 200 x 201 = 40,200. That is the sum of the *two* series.

40,200 ÷ 2 = 20,100. This was Carl Friedrich's answer for the sum of one series!
It is difficult to believe that he did not do most or all of this in his head.

1). Follow the above to find the sum of:

a) The first 100 whole numbers

b) The first 400 whole numbers

Were there any surprises? Look at all three **answers**, Carl Friedrich's and yours.
They are (or should be) Sum of first 100 = 5,050
Sum of first 200 = 20,100
Sum of first 400 = 80,200
Check your answers and correct mistakes, if any.

Sum of first 100 = 5,050
Sum of first 200 = 20,100
Sum of first 400 = 80,200

We see that 200 numbers brings a sum of almost (but not quite) *4 times* as much as the sum for 100 numbers. And 400 numbers brings a sum almost four times that for 200. Would you have expected each new result to be so much greater than the previous one?

**2a) Try finding the sum of the whole numbers from 1 to 800 mentally by following the pattern in the above results. _______________

b) Do the same for the sum of whole numbers from 1 to 1600. _________________

Perhaps you noticed in the box above, that the beginning digits of each answer is quadruple (4x) the beginning digits of the previous answer, but the ending digits are only twice the previous ending. The same should be true of your answers to exercise 2**:**

Answers: 2a) 320,400 **b)** 1,280,800

Mathematicians have been drawn to the study of series for many hundreds of years. If this were not so, mathematics could not have progressed to its present state. We have already met some series in Achilles and in Halfwalk. In later units we will see other important and interesting series.

In our Carl Friedrich Gauss work so far we have dealt only with series which start at 1 and increase by 1 each time.

3) Find the sum of these easy ones mentally, any way you can, and look for shortcuts.

a) 1 + 3 + 5 + 7 + 9 = ____ b) 45 + 50 + 55 = _____ c) 8 + 11 + 14 + 17 = ____

We wouldn't write out Gauss' procedure for these short problems, nor would Gauss himself, of course. The answer to b) is 50 x 3 = 150 because 50 can easily be seen as the average size term and there are 3 terms. Part a) is similar. When we know the average size of the terms we can pretend that each and every term is that size.

4) For 3a), (average term) x (number of terms) = ___ x ___ = _____

We see that in both 3a) and 3b) that the average of the terms is the middle term, or, it is the sum of the first and last divided by 2. (first + last) ÷ 2 = average term size.

Another way for 3a is to remember 1 + 3 + 5 + 7 + 9 as the sum of the <u>first 5 odd</u> numbers which we learned was $5^2 = 25$.

Part 3c) 8 + 11 + 14 + 17 = ____ is a little different. There is no middle term because there is an even number of terms, but we can easily see what the average is:

8 + 11 + 14 + 17

Pair the terms off and the average is halfway between either pair, with the middle pair the more convenient. (11 + 14)÷2 = 25/2. Number of terms = 4. 4 x $\frac{25}{2}$ = 50. Or, you could have seen the two 25's**:** 8 + 17, and 11 + 14 and found 50 by adding them. Even with a long series we can use the average times the number of terms.

$$1 + 2 + 3 + \ldots + 198 + 199 + 200$$

Average is $(1 + 200) \div 2 = \dfrac{201}{2} = 100\frac{1}{2}$. Awkward, isn't it. This would work, but the average is not a whole number. But wait!

If we leave the average as $\dfrac{201}{2}$ we could then easily multiply by 200 (number of terms).

$$\frac{201}{\underset{1}{\cancel{2}}} \times \frac{\overset{100}{\cancel{200}}}{1} = \frac{20{,}100}{1} = 20{,}100$$

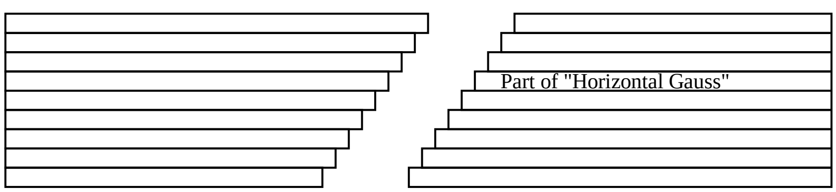
If you compare this to Gauss' work you will see that it is very similar to this.

So, to find the sum of a series, you only need to know 2 things:

1) The average of the terms

2) The number of terms.

Multiply them and you have it.



Believe it or not, finding the <u>number of terms</u> can require more thought than finding the average of the terms.

<u>Illustration 1</u>: How many terms are in this series?

$$130 + 132 + 134 + \ldots + 266 + 268 + 270$$

This is a good problem for using a technique called "<u>Reducing to a previous case</u>." This means to transform the problem into an *easier one* having *the same answer*. A first step would be to divide every number by 2 because they are all 2n numbers. Dividing each term by 2 does not change the number of terms but we do have a new series.

Do that and put the answer directly under the series below.

130 + 132 + 134 + . . . + 266 + 268 + 270

____

You now have an easier problem because the new numbers are *consecutive* but still have the same number of terms as the original. Your new series begins with 65 and ends with 135. Of course the new series has a different sum, but we don't care. It is only the number of terms we are interested in right now.

Another transformation is to subtract 64 from *each* term so that the new series will *start with* 1, but still have the original number of terms.

130 + 132 + 134 + . . . + 266 + 268 + 270

65 + 66 + 67 + . . . + 133 + 134 + 135

1 + 2 + 3 + . . . + 69 + 70 + 71

Now you know there are 71 terms in the series. We have again "reduced to a previous case" while preserving the number of terms. But we need the **original** series to find the average.

5) Here is a nice easy problem: first term 130 and last term 270 jumping by twos. Find the average mentally and then find the sum of the series: Not 36

(Average size) x (Number of terms)**:** ______ x _______ = _______

In the three rows of numbers above, you might argue that you could subtract 65 from 135 and get 70 terms. Close, but not right. 135 - 65 = 70. There is one term lost! So if you do use that idea, be sure to add 1 to your subtraction answer.

How many terms in 1 + 2 + 3? 3 – 1 = 2? Nope. There are **3** terms.

Illustration 2: Find the sum: 110 + 117 + 124 + . . . + 222 + 229

The key is first to find the length of each jump between numbers. Seeing that this is 7, you are then tempted to divide each number by seven for your first transformation. But, these are not 7n numbers. So change them to 7n numbers by subtracting (or adding) the correct amount to make each number a *multiple of seven*. Either subtract 5 from each term or add 2 to each term. Then you will be ready to find the number of terms in one more step.

6) What is the sum of the series 110 + 117 + 124 . . . + 222 + 229 ? __________

Exercises: Tell the average size of the terms in each series. Don’t find the sums.

7a) 1 + 2 + 3 + 4 + 5 ___

b) 1 + 2 + 3 + 4 + 5 + 6 _____

c) The whole numbers from 1 to 299 _____

d) The whole numbers from 1 to 300 ______

e) The whole numbers from 616 to 783 ______

f) First term is 20 and last is 16,000,000 ______

g) First term is 21 and last is 16,000,000 _______

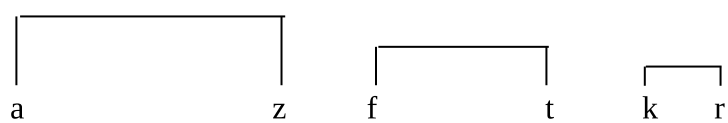
*h) First term is 11, second is 15, next to last is 231. _____

i) The terms on either side of the middle are 6.2 and 6.8 _____

j) The terms on either side of the middle are 1000 and 3000 _________

k) 23 + 30 + 37 + . . . + 93 _____

l) First term is 201 ¼, last term is 300 ¾ _________



m) a and z represent first and last terms and are 100 and 200 respectively _______

n) f and t are second and second-to-last: 57 and 73 respectively. _________.

o) k and r are third and third-to-last: 138.246 and 138.250 respectively (do mentally) _______

8) Find the number of terms in each of these. Don’t add them.

a) 14 + 15 + 16 + . . . + 37 + 38 _____

b) 3 + 6 + 9 + . . . + 36 _____ (Reduce to previous case.)

c) 23 + 30 + 37 + . . . + 93 _____

*d) The whole numbers from 0 to 100, inclusive _____

*e) The multiples of 3 between 100 and 200 _____

9) Find the sums: Calculator OK.

a) The whole numbers from 1 to 76. _______

b) 13 + 18 + 23 + . . . + 73 + 78 _______

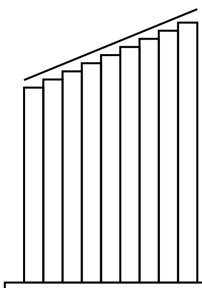
c) 12 + 16 + 20 + . . . + 108 _______

CARL FRIEDRICH GAUSS TEACHES THE TEACHER **Unit 22

d) 204 + 212 + . . . + 396 _________

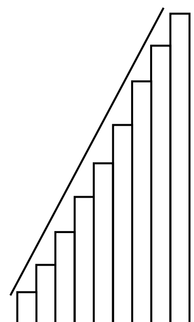
*e) The first 5000 odd numbers. _____________________

Fig. A



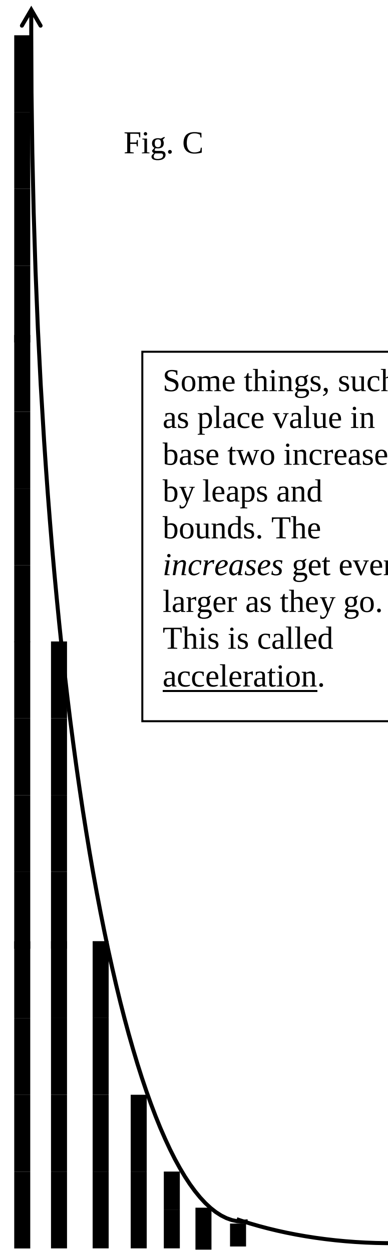
Some things grow rather slowly.
10+11+12 . .

Fig. B



Some things grow quickly
3+5+7+9+. .

Fig. C



Some things, such as place value in base two increase by leaps and bounds. The *increases* get ever larger as they go. This is called <u>acceleration</u>.

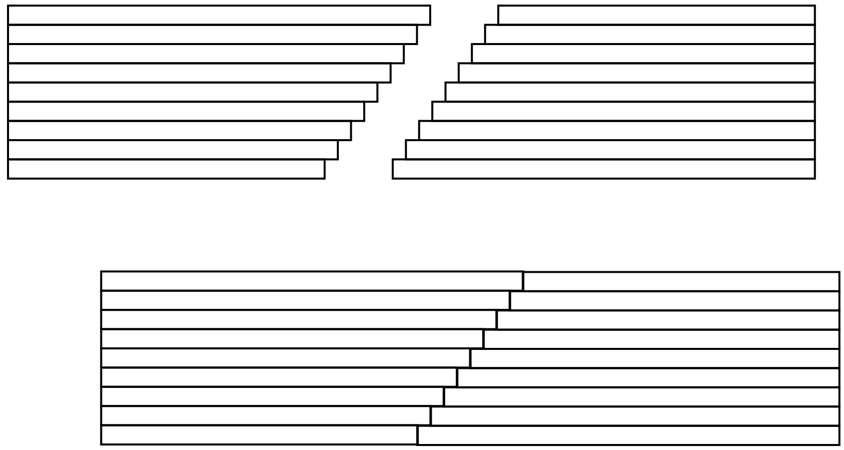
Figure C could be a picture of some things you have studied, such as place value in base two, and if you slide *down the curve*, you experience how the size of the stages in "Halfroom" get smaller and smaller, with each stage getting smaller by becoming half of the preceding stage. While the other graphs are clearly increasing without bound or limit, this downward curve is flattening out and approaches ever closer to a horizontal position. The little black bars are getting very tiny and are approaching zero as a bound or limit.

10) **On which figure would Gauss' method, or the average method, <u>not</u> work?** ______

An account of Gauss' youth can be found in
Men of Mathematics by Eric Temple Bell, 1937

CARL FRIEDRICH GAUSS TEACHES THE TEACHER **Unit 22

11) The design from page 5 is shown here and shown again moved together. If the shortest original rod was 10 and the longest was 18, what is the length of each joined rod? ____ The width of each rod is 1. What is the amount of area covered by the joined rods?



_______(Square units) What is the area of one set of original rods? ______(Sq.. units)

You might have guessed that this is a diagram model of Carl Friedrich Gauss' solution.

Answers

1a) 5,050 **b)** 80,200

2a) 320,400 **b)** 1,280,800

3a) 25 **b)** 150 **c)** 50

4) 5 x 5 = 25

5) 71 x (200/2) = 14,200

6) 3,051

7a) 3 **b)** 3 ½ ***c)** 150 **d)** 150 ½ **e)** 699.5

f) 8,000,010 **g)** 8000010.5 **h)** 123 **i)** 6.5 **j)** 2000

k) 58 **l)** 251 **m)** 150 **n)** 65 **o)** 138.248

8a) 25 **b)** 12 **c)** 11 **d)** 101 ***e)** 33

9a) 2926 **b)** 637 **c)** 1500 **d)** 7500 **e)** Remember the shortcut? $5000^2 = 25{,}000{,}000$

10) Fig. C

11) 28; 9 x 28 = <u>252</u>; 252 ÷ 2 = <u>126</u>. These calculations for the diagram are similar to what Gauss would have done without the diagram. An account of Gauss' early years is given in the book "Men of Mathematics" by Eric T.. Bell, 1937.

Order of Operations — Unit 23

These look easy. Do most mentally, writing just the final answer.

1) $10 - 3 \times 2 =$

2) $6 + 2 \times 5 =$

3) $8 \div 2 - 3 =$

4) $16 - 8 \div 4 =$

5) $(8 - 6) \div 2 =$

6) $6 - 2 \times 3 =$

7) $8 \times 4 - 3 =$

8) $3(6 - 2) =$

9) $3 \times 4^2 =$

10) $\frac{1}{2}(4 + 8) =$

11) $10 - (4 - 2) =$

12) $6 - 2^2 =$

13) $(6 - 2)^2 =$

14) $3(8 - 3)^2 =$

15) $\frac{10+2}{4} - 3 =$

16) $10 - \frac{5+3}{2} =$

The likely *wrong* answer to exercise 1 is 14 and to exercise 2 is 40. Those incorrect answers show the tendency to do a string of operations from left to right because that's the way we read. If instead you got an answer of 4 for exercise 1 and 16 for exercise 2, congratulations. In mathematics it has been agreed that multiplications and divisions will be done *before* additions and subtractions, no matter what the order of appearance is.

You probably got an answer of 1 for exercise 3 and that is correct. Go back and fix your answer to exercise 4 now that you know that multiplications and divisions are done before additions or subtractions. The answer should be 14, not 2.

Answers

1) 4 **2)** 16 **3)** 1 **4)** 14 **5)** 1 **6)** 0 **7)** 29 **8)** 12 **9)** 48 **10)** 6
11) 8 **12)** 2 **13)** 16 **14)** 75 **15)** 0 **16)** 6

ORDER OF OPERATIONS Unit 23

There are other things to be careful of. In exercise 8, 3(6 – 2), we find parentheses. These are grouping symbols. Also, there is no operation symbol telling you what to do with the 3 in exercise 8. Agreement says to leave out the operation sign when multiplication is intended. So, 3(6 – 2) means first do the grouped (6 – 2) and then multiply by 3. Of course a number like 56 means fifty-six, not 5x6. The multiplication sign is left out only when the meaning is clear. This comes up again in exercises 10 and 14.

It is also true that the numerator (or denominator) of a fraction which contains any addition or subtraction should be combined into a single number before trying to operate with the fraction. Thus, in $\frac{10+2}{4}$ - 3, we add the 10 + 2 first and then proceed to divide by 4 and finally, subtract 3. Perhaps you got exercise 15 right anyway.

One more thing**:** exponents.

When an exponent is used with a number, that number should be raised to the power indicated before operating with another number.

In 3×4^2 for instance, the exponent applies to the 4 only. Thus, it means 3 x 16, not 12^2.
Use $3(2^3 - 3)^2$ to refer to and to study the following summary of what we have said:

> First, combine numbers inside the parentheses; this first one is often not listed but left to common sense.
> Second, apply exponents.
> Third, perform multiplications and divisions.
> Fourth, perform additions and subtractions.
> The final result is _________.

It is easy to see that we must have agreement on the order of operations or we would not all get the same answer to the same problem, which, for $3(2^3 - 3)^2$, above, is 75. Recall that $\sqrt{64} = 8$, $\sqrt{100} = 10$

ORDER OF OPERATIONS | Unit 23

Do these, referring to the rules on pages 1 - 2 if you need to:

17) $16 - 4 \times 2 =$

18) $8 + 2 \times 3 =$

19) $14 \div 7 - 2 =$

20) $25 - 15 \div 5 =$

21) $\frac{10-6}{4} + 3 =$

22) $20 - 8 \times 2 =$

23) $6 \times 4 - 3 =$

24) $6 \times 3^2 =$

25) $(6 \times 3)^2 =$

26) $16 - (7 - 2) =$

27) $12 - 3^2 =$

28) $(12 - 3)^2 =$

29) $6(5 - 2)^2 =$

30) $\frac{8-3}{5} + 1 =$

31) $24 - \frac{6+2}{8} =$

32) $5(5 - 3)^3 =$

33) $\frac{1}{3}(6 + 9) =$

34) $3 \times \sqrt{16} + 2^3 =$

35) $3 + \sqrt{16} \times 2^3 =$

Now the focus of the work changes to *organization*. This becomes more and more important as you go on in mathematics. **Examples for study:**

$4(3 + 7) + 3(5 + 2) =$

$4 \times 10 + 3 \times 7 =$

$40 + 21 = 61.$

Don't say "why bother"?
Get better organizing
habits now.
It matters.

$8(3 + 2)^2 - 5(6 - 4)^3 =$

$8 \times 5^2 - 5 \times 2^3 =$

$8 \times 25 - 5 \times 8 =$

$200 - 40 = 160$

Work gradually, perhaps more gradually than the rules for order of operation require.

36) $6(3 + 2) + 5(4 - 2) =$

37) $(5 - 2)^2 - (3 - 1)^2 =$

ORDER OF OPERATIONS Unit 23

Keeping work orderly is not only for reasonably neat appearance, but also for reducing the number of errors that are made by carrying too much in the head.

38) $9(6 + 1) - 8(4 - 1) =$

39) $3(3 + 2)^2 + 2(5 - 2)^2 =$

40) $12\left(\frac{1}{2}\right)^2 + 3(1 + 2^2)^2 =$

Note: Different sizes in parentheses are not usual. They are used below as an aid. Work from the inside out. Use the exercise below left as a guide for exercise 40. Notice that each left parenthesis is matched by a right parenthesis at every stage of the problem solution.

$\Big(2\sqrt{16} - 3(4^2 - 5 \times 3)\Big)^3 =$

$\Big(2 \times 4 - 3(16 - 15)\Big)^3 =$

$(8 - 3 \times 1)^3 =$

$5^3 = 125$

41) $18\left(\frac{2}{3}\right)^2 + 3(1500 - 1498)^2 =$

42) $\left(\frac{8-2}{3}\right)^3 + 2(8-5)^2 =$

43) $(3\sqrt{25} + 2(2^4 - 3 \times 4))^2 =$

There is at least one multiplication which you will not want to do in your head. In exercise 43, do it (or them) off to one side neatly. There is no need to hide it.

*44) $\Big(\{(9 - 2) + 10\}^2 + \{(6 + 2)^2 - 9 \times 7\}\Big)^2 =$

ORDER OF OPERATIONS Unit 23

Answers

17) 8	**18)** 14	**19)** 0	**20)** 22
21) 4	**22)** 4	**23)** 21	**24)** 54
25) 324	**26)** 11	**27)** 3	**28)** 81
29) 54	**30)** 2	**31)** 23	**32)** 40
33) 5	**34)** 20	**35)** 35	**36)** 40
37) 5	**38)** 39	**39)** 93	**40)** 78
41) 20	**42)** 26	**43)** 529	**44)** 84,100

Suggested solutions to exercise 43:

$$\left(\{(9-2)+10\}^2 + \{(6+2)^2 - 9 \times 7\}\right)^2 =$$

$$\left(\{7+10\}^2 + \{8^2 - 63\}\right) =$$

$$(289+1)^2 =$$

$$290^2 = 84{,}100$$

or

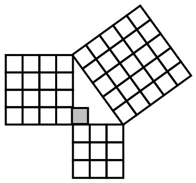
$$\left(\{(9-2)+10\}^2 + \{(6+2)^2 - 9 \times 7\}\right)^2 =$$

$$\left(\{17\}^2 + \{64-63\}\right)^2 =$$

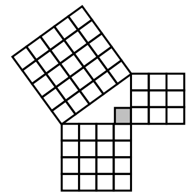
$$290^2 = 84{,}100$$

PYTHAGORAS Unit 24

(Pythagoras Rhymes with "Thy flag o'er us")

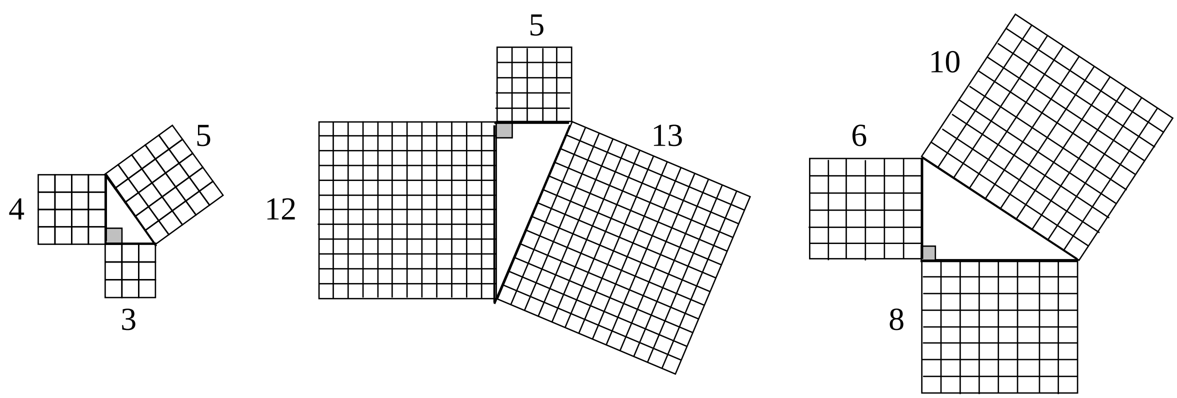


"I'm very well acquainted too with matters mathematical,
I undersand equations, both the simple and quadratical, About
binomial theorem I'm teeming with a lot of news, With many
cheerful facts about the square of the hypotenuse."
(From the Operetta "Pirates of Penzance" by Gilbert and Sullivan)



The ancient Greek named Pythagoras (580-500 B.C.) is famous today for proving what is known as the Pythagorean Theorem. Greeks of his day were influenced by Pythagoras' passion for numbers and many wanted to join his secret society. Pythagoras believed that numbers were the actual atoms that made up the physical world. Number, therefore, would yield most of the secrets about all of creation, in his view. Most of Pythagoras' ideas remained secret until after his death. The theorem which bears his name is illustrated in the figures below ("theorem" means a mathematical statement that can be proved).

Note: If you are very familiar with solving problems using the Pythagorean Theorem, then skip to example 8 and go on from there.

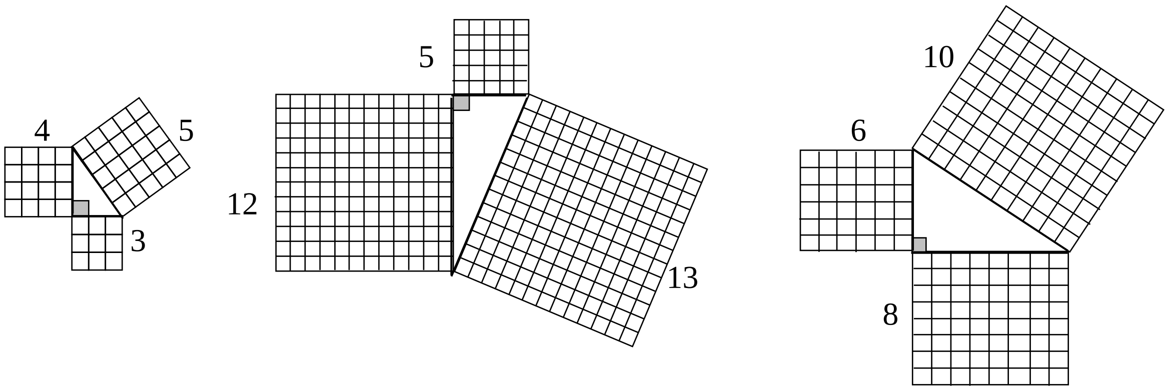


The numbers above are whole numbers so we call them Pythagorean triplets.

The three diagrams have several things in common:

- Each contains a right triangle, a triangle with a right angle (90°).
- Each contains a square of squares on each side of its triangle.
- For each right triangle, the number of little squares in the largest square, (25 in the first diagram), is equal to the number of little squares in the other two squares added together. 9 + 16 = 25 in the first diagram.

PYTHAGORAS Unit 24



In the second diagram, , $5^2 + 12^2 = 13^2$

That is, $25 + 144 = 169$

and $169 = 169$

> Now it is easy to see why 5^2 is called "five square".

1) Follow the form above for (6, 8, 10):

6^2 +___--- = ____---

36 + ___ = ____

____ = ____

Answers are in next unit.

2) Using the same form, verify whether

8, 15, 17 is a Pylthagorean triplet

___--- + ___--- = ___---

____ + ____ = ____

_____ = ____

Example: Verify whether 4, 5, 6 is a Pythagorean triplet:

$4^2 + 5^2 = 6^2$

$16 + 25 = 36$

$41 = 36$ No. Or, better yet, $41 \neq 36$ (does not equal).

Use the form and verify whether the following are Pythagorean triplets:

3) 5, 6, 7

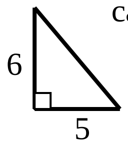
4) 7, 24, 25

5) 30, 40, 50 (Multiples of 3,4,5)

6) 8, 14, 17

PYTHAGORAS Unit 24

You discovered in exercise 3 that 5, 6, 7 is not a Pythagorean triplet. Does this mean 5 and 6 cannot be the values of the two smallest sides of a right triangle? Of course not. But it does mean, that if the legs (smallest two sides) are 5 and 6, then the longest side is not 7. The longest side of a right triangle is called hypotenuse (rhymes with "Thy cotton noose").



Since $5^2 + 6^2 \neq 7^2$, what is the length of the hypotenuse?
"Does not equal"

The problem can be stated this way: $5^2 + 6^2 = \boxed{\ }^2$

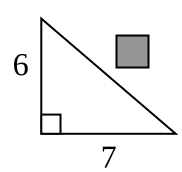
That states the problem, but it does not tell us what, if anything, can be put in the box to make the statement true. Let's continue with the statement.

$$5^2 + 6^2 = \boxed{\ }^2$$
$$25 + 36 = \boxed{\ }^2$$
$$61 = \boxed{\ }^2$$

Now the problem is stated so that we need to put a number in the box which, when squared, will give 61. We can see right away that the answer cannot be a whole number, because $7^2 = 49$ and $8^2 = 64$; 61 lies between 49 and 64.

7a) Between what two whole numbers is the length of the hypotenuse we are looking for? ____ and ____ (Not 49 nor 64)

b) Following the above form, use the numbers 6, 7, $\boxed{\ }$, and find between what two whole numbers the hypotenuse lies.



Your answer: Between ____ and ____

Check all answers to here: Answers found in Unit 25: PYTHAGORAS ANSWERS

PYTHAGORAS Unit 24

Time out from Pythagoras. No calculator unless specified.

8) You are probably familiar with the square root symbol, $\sqrt{\ }$: $\sqrt{64} = 8$, $\sqrt{25} = 5$

a) $\sqrt{100} =$ ___ b) $\sqrt{121} =$ ___ c) $\sqrt{\sqrt{81}} =$ ___ d) $\sqrt{400} =$ ___ e) $\sqrt{\boxed{}} = 6$

f) $\sqrt{\boxed{}} = 100$ (Not 10) g) $\sqrt{\sqrt{\boxed{}}} = 2$

Note: A "perfect square" number is one whose square root is an exact whole number.

h) The whole number nearest to $\sqrt{50}$ is ____. i) The perfect square nearest 50 is ____.

j) $\sqrt{116}$ lies between what two whole numbers: ____ and ____.

k) T/F $\sqrt{36} = \sqrt{9} \times \sqrt{4}$ ____ l) T/F $\sqrt{64} = \sqrt{16} \times \sqrt{4}$ ____.

m) $\sqrt{100} \times \sqrt{100} =$ _____ n) $\sqrt{121} \times \sqrt{121} =$ _____

Check all answers to exercise 8: **Answers found in Unit 25: PYTHAGORAS ANSWERS**

9a) $\sqrt{4} \times \sqrt{4} =$ ___ b) $\sqrt{17} \times \sqrt{17} =$ ___ c) $\sqrt{85} \times \sqrt{85} =$ ___ d) $\left(\sqrt{9}\right)^2 =$ ___

e) $\left(\sqrt{11}\right)^2 =$ ___ f) $16 = \boxed{}^2$ g) $17 = \boxed{}^2$ (see part b) h) $85 = \boxed{}^2$

It is important to understand what was happening in exercise 9. In f), 16 is a perfect square, so 4 can be put in the box (although you might have put the cumbersome $\sqrt{16}$). But in exercise g and h, you <u>need</u> to use $\sqrt{17}$ and $\sqrt{85}$ in the boxes to express the "exact" answers. It might seem strange to use $\sqrt{17}$ and $\sqrt{85}$ this way. It's saying "The number which is squared to get 17 is $\sqrt{17}$ ". Strange, but O.K., and very useful, as we'll see later.

10a) $\sqrt{92} \times \sqrt{92} =$ ____ b) $\left(\sqrt{92}\right)^2 =$ ____ c) $\sqrt{92 \times 92} =$ ____ d) $\sqrt{92^2} =$ ____

e) $\sqrt{92}$ names the number which, when squared, gives ____.

Check answers to exercise 9 and 10: **Answers found in Unit 25: PYTHAGORAS ANSWERS**

11) No Calculator for exercise a - f

a) (T/F) $\sqrt{17} \approx 4.1$ ($\approx$ means approximately equal) ___

b) Yes/No Is 4.1^2 exactly equal to 17? _____

c) 4.1^2 = ________ (multiply it out)

d) 4.2^2 = ________

e) Which is closer to 17, 4.1^2 or 4.2^2 ? _______

f) Is there any one-decimal value closer to $\sqrt{17}$ than 4.1? (Y/N) _____

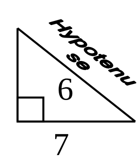
g) Use $\boxed{\sqrt{}}$ on your calculator and write the value for $\sqrt{17}$ __________________

h) With exercise g) answer in the window of your calculator, press $\boxed{x^2}$: ____

i) Do you think the calculator's result in g) is exactly right? (Y/N) ____

j) Do you think the calculator's result in h) is exactly right? (Y/N) ____

Check answers to exercise 11: Answers found in Unit 25: PYTHAGORAS ANSWERS



Time out is over as we return to $6^2 + 7^2 = \boxed{}^2$ from exercise 7b.

12) No calculator $36 + 49 = \boxed{}^2$

You might want to linger over a) and b). It's simple, but not easy.

a) $85 = (\quad)^2$

b) Exactly, what number squared equals 85? ______

c) Calculator okay. To the nearest tenth, how long is the hypotenuse in part b)? _____

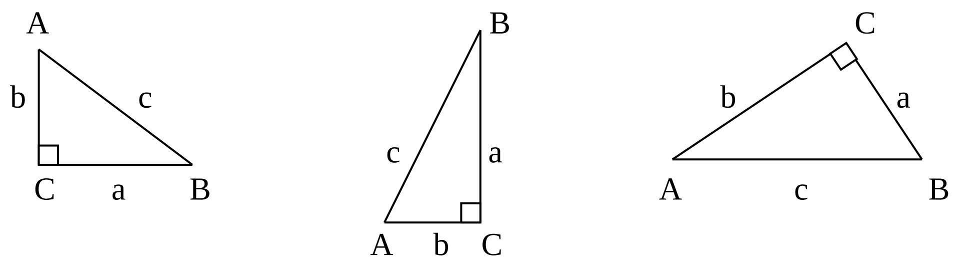
Talking about eight or ten decimal places on a calculator for measures is not really practical. Also, exact measures are nonexistent. There is always some error in measurement, no matter how small. "Error" in this case does not mean mistake. It means that no measuring instrument can be perfect, nor can the thing measured.

Check answers: Answers found in Unit 25: PYTHAGORAS ANSWERS

PYTHAGORAS Unit 24

The technology of the hand held calculator far outstrips our ability to measure line segments, but we can still use numbers like $\sqrt{85} = 9.219544457$ when treating them as something other than measures.

Now that you understand what is needed in solving Pythagoras problems, we are going to use a more customary form and procedure very much like what we have been using.



The above lettering is frequently used but not always. Notice that C names the right angle, and A names one of the other angles with B naming the remaining angle. The side opposite $\angle C$ (angle C) is the hypotenuse. When the lettering A, B, C, *are* used, c is usually the hypotenuse, and $\angle A$ and $\angle B$ are usually the acute (less than 90º) angles. Side a is opposite $\angle A$ and side b is opposite $\angle B$. Sides a and b are the legs of the right triangle.

The Pythagorean formula is $a^2 + b^2 = c^2$

Always begin by writing the formula and then substituting the given numbers for the letters.

Sample: Find the length of the hypotenuse in a right triangle where a = 9 in. and b = 12 in.

$$a^2 + b^2 = c^2$$
$$9^2 + 12^2 = c^2$$
$$81 + 144 = c^2$$
$$225 = c^2$$
$$c = \sqrt{225}$$
$$c = 25 \text{ in.}$$

1. Write the formula.
2. Substitute the numbers.
3. Do the computation.
4. Note that the order is switched at this point. It's helpful.
5. Complete the computation, round to nearest tenth (usually) if answer isn't a whole number.

PYTHAGORAS Unit 24

$a^2 + b^2 = c^2$

Calculator okay on this page and the next.

13) A right triangle has legs of 4 cm and 5 cm. To the nearest tenth cm, how long is the hypotenuse? Here is a start. Complete it.

$a^2 + b^2 = c^2$

$4^2 + 5^2 = c^2$

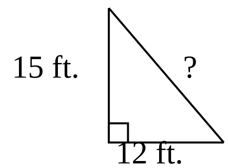
___ + ___ $= c^2$

______ $= c^2$

(Order switched) ___ $= \sqrt{41}$

Calculator here ___ = _____cm.

14) Find the hypotenuse. Continue to show formula and work. $a^2 + b^2 = c^2$ etc.



15) The formula takes a new look. Suppose the hypotenuse and one leg are given and you must find the other leg.

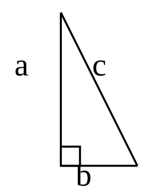
Since $a^2 + b^2 = c^2$,

then $a^2 = c^2 – b^2$

and $b^2 =$ ___ - ___

Since 5 + 8 = 13,
Then 5 = 13 – 8
And 8 = 13 - 5

16)



a = 24 in.
b = ?
c = 25 in.

$a^2 + b^2 = c^2$

$b^2 = c^2 – a^2$ Note change because b is not given

$b^2 =$ ___ $- 24^2$

$b^2 =$ ____ - 576

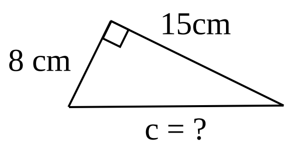
$b^2 =$ _________

$b = \sqrt{\quad}$

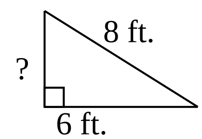
b = ____

PYTHAGORAS Unit 24

17) Use formula and form (always).



18)



19) $5^2 + \left(\sqrt{11}\right)^2 = c^2$

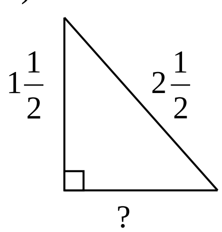
a = 5

b = ____

c = ? (finish it)

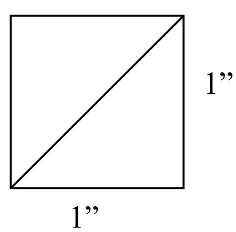
20) a = 10, b = 12, find c.

21)



Hint: Use decimals. Also note that a leg is missing, not the hypotenuse.

22) Find the length of the diagonal of a square whose side is 1 inch, to the nearest tenth inch.



As you meet Pythagoras problems in later work you can leave out some of the steps insisted on here. Use your judgement about what and how much to leave out. The more you skip, the greater the chance for error *and* the greater the difficulty in finding errors.

On the other hand, as your work matures, you *should* seek levels of ease with letting procedures get briefer but not too brief.

Answers found in Unit 25: PYTHAGORAS ANSWERS

PYTHAGORAS ANSWERS Unit 25

1) Follow the form above for (6, 8, 10):

$6^2 + \mathbf{8^2} = \mathbf{10^2}$

$36 + \mathbf{64} = \mathbf{100}$

$\mathbf{100} = \mathbf{100}$

2) Using the same form, verify whether 8, 15, 17 is a Pythagorean triplet

$\mathbf{8^2 + 15^2 = 17^2}$

$\mathbf{64 + 225 = 289}$

$\mathbf{289 = 289}$

3) 5, 6, 7

$\mathbf{5^2 + 6^2 = 7^2}$

$\mathbf{25 + 36 = 49}$

$\mathbf{61 \neq 49}$ **NO**

4) 7, 24, 25

$\mathbf{7^2 + 24^2 = 25^2}$

$\mathbf{49 + 576 = 625}$

$\mathbf{625 = 625}$ **YES**

5) 30, 40, 50 (Multiples of 3,4,5)

$\mathbf{30^2 + 40^2 = 50^2}$

$\mathbf{900 + 1600 = 2500}$

$\mathbf{2500 = 2500}$ **YES**

6) 8, 14, 17,

$\mathbf{8^2 + 14^2 = 17^2}$

$\mathbf{64 + 196 = 289}$

$\mathbf{260 \neq 289}$ **NO**

$5^2 + 6^2 = \blacksquare^2$

$25 + 36 = \blacksquare^2$

$61 = \blacksquare^2$

Now the problem is that we need to put a number in the box which, when squared, will give 61. We can see right away that the answer cannot be a whole number, because $7^2 = 49$ and $8^2 = 64$. <u>61 lies between 49 and 64</u>.

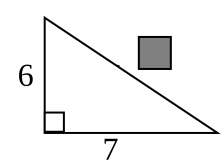
7a) Between what two whole numbers is the length of the hypotenuse we are looking for? **7** and **8** (Not 49 nor 64)

b) Follow the above form for 6, 7, $\square$ and find between what two whole numbers the hypotenuse lies.

$\mathbf{6^2 + 7^2 = \blacksquare^2}$

$\mathbf{36 + 49 = \blacksquare^2}$

$\mathbf{85 = \blacksquare^2}$



Your answer: Between **9** and **10** because 85 lies between 81 and 100.

Go back to top of page 4, Pythagoras

Pythagoras Answers Unit 25

8a) $\sqrt{100}=\mathbf{10}$ **b)** $\sqrt{121}=11$ **c)** $\sqrt{\sqrt{81}}=3$ **d)** $\sqrt{400}=20$ **e)** 36

f) $\sqrt{10{,}000}$ = 100 (Not 10) **g)** $\sqrt{\sqrt{16}}$ = 2

Note: A "perfect square" number is one whose square root is an exact whole number.

h) The whole number nearest to $\sqrt{50}$ is 7 **i)** The perfect square nearest 50 is 49.

j) $\sqrt{116}$ lies between what two whole numbers: 10 and 11 **.**

k) T/F $\sqrt{36}=\sqrt{9}\times\sqrt{4}$ T **l)** T/F $\sqrt{64}=\sqrt{16}\times\sqrt{4}$ T

m) $\sqrt{100}\times\sqrt{100} = 100$ **n)** $\sqrt{121}\times\sqrt{121} = 121$

Return to Unit 24: Pythagoras.

9a) $\sqrt{4}\times\sqrt{4} = 4$ **b)** $\sqrt{17}\times\sqrt{17} = 17$ **c)** $\sqrt{85}\times\sqrt{85} = 85$ **d)** $\left(\sqrt{9}\right)^2 = 9$

e) $\left(\sqrt{11}\right)^2 = 11$ f) $16 = 4^2$ **g)** $17 = \left(\sqrt{17}\right)^2$ (see part b) **h)** $85 = \left(\sqrt{85}\right)^2$

10a) $\sqrt{92}\times\sqrt{92} = 92$ **b)** $\left(\sqrt{92}\right)^2 = 92$ **c)** $\sqrt{92\times 92} = 92$ **d)** $\sqrt{92^2} = 92$

e) $\sqrt{92}$ names the number which, when squared, gives 92.

Return to Unit 24: Pythagoras.

11a) T/F $\sqrt{17}\approx 4.1$ (≈ means approximately equal) T

b) Yes/No Is 4.1^2 exactly equal to 17? NO

c) $4.1^2 = 16.81$ **d)** $4.2^2 = 17.64$ **e)** Which is closer to 17, 4.1^2 or 4.2^2 ? 4.1^2

f) Is there any one-decimal value closer to $\sqrt{17}$ than 4.1? (Y/N) NO

g) Use $\boxed{\sqrt{\ }}$ on your calculator and write its value for $\sqrt{17}$ 4.1231056(26)

h) With example g) answer in the window of your calculator, press $\boxed{x^2}$: 17

i) Do you think the calculator's result in g) is exactly right? (Y/N) No

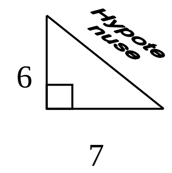
j) Do you think the calculator's result in h) is exactly right? (Y/N) No

Return to Unit 24: Pythagoras.

PYTHAGORAS ANSWERS Unit 25

Time out is over as we return to $6^2 + 7^2 = \boxed{}^2$ from exercise 7.

$36 + 49 = \boxed{}^2$



12a) $85 = \left(\sqrt{85}\right)^2$

b) Exactly, what number squared = 85 ? $\sqrt{85}$

c) To the nearest tenth, how long is the hypotenuse in part a)? 9.2

Return to Unit 24: Pythagoras.

13) A right triangle has legs of 4 cm and 5 cm. To the nearest tenth cm, how long is the hypotenuse? Here is a start. Complete it.

$a^2 + b^2 = c^2$

$4^2 + 5^2 = c^2$

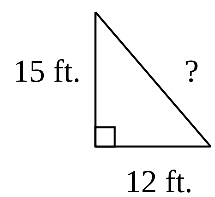
$16 + 25 = c^2$

$41 = c^2$

$c = \sqrt{41}$

$c = 6.4$ cm

14) Find the hypotenuse. Continue to show formula and work.



$a^2 + b^2 = c^2$ etc.

$15^2 + 12^2 = c^2$

$225 + 144 = c^2$

$369 = c^2$

$c = \sqrt{369}$

$c = 19.2$

PYTHAGORAS ANSWERS — Unit 25

15) The formula takes a new look. Suppose the hypotenuse and one leg are given and you must find the other leg.

Since $a^2 + b^2 = c^2$,

then $a^2 = c^2 – b^2$

and $b^2 = c^2 - a^2$

Since 5 + 8 = 13,

Then 5 = 13 – 8

And 8 = 13 - 5

16) a c

a = 24 in.
b = ?
c = 25 in.

b

$a^2 + b^2 = c^2$

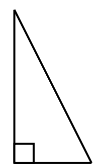
$b^2 = c^2 – a^2$

Note change because b is not given

$b^2 = 25^2 - 24^2$

$b^2 = 625 - 576$

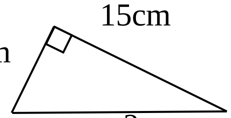
$b^2 = 49$



$b = \sqrt{49}$

$b = 7$ in.

17) Use formula and form (always).



.

$a^2 + b^2 = c^2$

$8^2 + 15^2 = c^2$

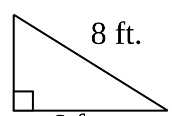
$64 + 225 = c^2$

$289 = c^2$.

$c = \sqrt{289}$

c = 17 cm.

18)



$a^2 + b^2 = c^2$

$a^2 = c^2 – b^2$

$a^2 = 8^2 - 6^2$

$a^2 = 64 - 36$

$a^2 = 28$

$a = \sqrt{28}$

a = 5.3 ft.

PYTHAGORAS ANSWERS

Unit 25

19) a = 5

b = $\sqrt{11}$

c = ?

$$5^2 + \left(\sqrt{11}\right)^2 = c^2$$

$$25 + 11 = c^2$$

$$36 = c^2$$

$$c = \sqrt{36}$$

$$c = 6$$

20) a = 10, b = 12, find c.

$$a^2 + b^2 = c^2$$

$$10^2 + 12^2 = c^2$$

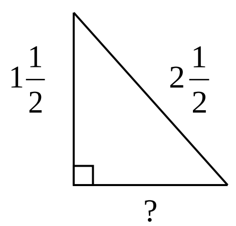
$$100 + 144 = c^2$$

$$244 = c^2$$

$$c = \sqrt{244}$$

$$c = 15.6$$

21)



Hint: Use decimals. Also note that a leg is missing, not the hypotenuse.

$$a^2 + b^2 = c^2$$

$$b^2 = c^2 - a^2$$

$$b^2 = 2.5^2 - 1.5^2$$

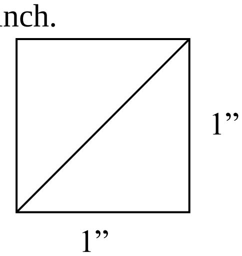
$$b^2 = 6.25 - 2.25$$

$$b^2 = 4$$

$$b = \sqrt{4}$$

$$b = 2$$

22) Find the length of the diagonal of a square whose side is 1 inch, to the nearest tenth inch.



$$a^2 + b^2 = c^2$$

$$1^2 + 1^2 = c^2$$

$$1 + 1 = c^2$$

$$2 = c^2$$

$$c = \sqrt{2}$$

$$c = 1.4 \text{ in}$$

MULTIPLYING BY 12

Unit 26

1) Some easy ones: 2 x 12 = ____ 4 x 12 = ____ 3 x 12 = ____ 9 x 12 = _ _ 8 ← Endings help

2) Now look at these:

5 x 12 = 60

6 x 12 = 72

7 x 12 = 84

See a pattern?

8 x 12 = _ 6

You do it

9 x 12 = _ _ 8

3) Cover up 2) and do these:

7 x 12 = _ 4
8 x 12 = _ 6
6 x 12 = _ 2
9 x 12 = _ _ 8
5 x 12 = _ 0

4) Here is how to remember the *last* digit:

7 x 2 = ___, so 7 x 12 must end with 4.
The *last* digit for 8 x 12 is ___.
The last for 6 x 12 is ___, for
9 x 12 is __, and for 5 x 12 is __.

5) Reminders:

9 x 12 = _ _ 8
7 x 12 = _ 4
6 x 12 = 7 _
8 x 12 = 9 _
5 x 12 = _ 0

6) Without looking back, do these:

5 x 12 = _ _

6 x 12 = _ _

7 x 12 = _ _

8 x 12 = _ _

9 x 12 = _ _ _

7) And now again:

8 x 12 = ___

6 x 12 = ___

7 x 12 = ___

5 x 12 = ___

9 x 12 = ___

8) Do these quickl

8 x 12 = ____	12 x 8 = ____	9 x 12 =	
12 x 7 = ____	6 x 12 = ____	12 x 5 = ____	
12 x 6 = ____	5 x 12 = ____	7 x 12 = ____	12 x 9 = ____

9) For 11 x 12 picture only the addition part of the multiplication.
Or, just memorize: 11 x 12 = 132.

12 x 12 = 144
That's 12 dozen or one gross.
12^2 = 144 seems easy to recall.

```
  12
 x11
  12
 12
 132
```

10) Here they are all mixed together. Try not to look back.

10) 8 x 12 = ___ 1 x 12 = ___ 12 x 7 = ___ 12 x 5 = ___ 12 x 9 = ___ 5 x 12 = ___

12^2 = ___ 7 x 12 = ___ 12 x 6 = ___ 9 x 12 = ___ 11x 12 = ___

3 x 12 = ___ 12 x 4 = ___ 0 x 12 = ___ 6 x 12 = ___ 10 x 12 = ___

MULTIPLYING BY 12 Unit 26

Answers

1) Some easy ones: 2 x 12 = 24   4 x 12 = 48   3 x 12 = 36   9 x 12 = 108

2) Now look at these:

5 x 12 = 60
6 x 12 = 72
7 x 12 = 84
See a pattern?
8 x 12 = 96
You do it
9 x 12 = 108

3) Cover up 2) and do these:
7 x 12 = 84
8 x 12 = 96
6 x 12 =72
9 x 12 =108
5 x 12 = 60

4) Here is how to remember the *last* digit:

7 x 2 =14, so 7 x 12 must end with 4.
The *last* digit for 8 x 12 is 6.
The last for 6 x 12 is 2, for
9 x 12 is 8, and for 5 x 12 is 0.

5) Reminders:

9 x 12 = 108
7 x 12 = 84
6 x 12 = 72
8 x 12 = 96
5 x 12 = 60

6) Without looking back, do these:
5 x 12 = 60
6 x 12 = 72
7 x 12 = 84
8 x 12 = 96
9 x 12 = 108

7) And now again:
8 x 12 = 96
6 x 12 = 72
7 x 12 = 84
5 x 12 = 60
9 x 12 = 108

8) Do these quickl

8 x 12 = 96   12 x 8 = 96   9 x 12 = 108
12 x 7 =84   6 x 12 = 72   12 x 5 = 60
12 x 6 = 72   5 x 12 = 60   7 x 12 = 84   12 x 9 = 108

9) For 11 x 12 picture only the addition part of the multiplication. Or, just memorize it. 11 x 12 =132.

$$\begin{array}{r} 12 \\ \underline{\times 11} \\ 12 \\ \underline{12} \\ 132 \end{array}$$

12 x 12 = 144
That's 12 dozen or one gross.
12^2 = 144 seems easy to recall.

10) Here they are all mixed together. Try not to look back.

10) 8 x 12 = 96   1 x 12 = 12   12 x 7 = 84   12 x 5 = 60   12 x 9 = 108   5 x 12 = 60
12^2 = 144   7 x 12 = 84   12 x 6 = 72   9 x 12 = 108   11 x 12 = 132
3 x 12 =36   12 x 4 = 48   0 x 12 = 0   6 x 12 = 72   10 x 12 = 120

SOME NEW, SOME OLD, FRACTIONS, DECIMALS AND PERCENTS — *Unit 27

All Work Mental.

This unit touches and extends some of the highlights of fractions, decimals and percents. It is review for those who have studied the topics in their pre-algebra work. It is not comprehensive. Most should be done mentally. Think. It helps.

1a) $\frac{1}{2} = \_\_\_\% = \frac{\square}{100} = .\_\_ \_\_ = .5$ b) $\frac{1}{4} = \_\_\_\% = \frac{\square}{100} = .\_\_ \_\_$

2a) $\frac{1}{2}$ of $\frac{1}{4} = \frac{1}{\square} = \_\_\_\% = \frac{\text{⯃}}{100} = .12\frac{1}{2}$ **;** 1/2 of 25**% = _____%**

2b) $\frac{1}{8} = \_\_\_\_\_\%$, $\frac{2}{8} = \_\_\_\_\%$, $\frac{3}{8} = \_\_\_\_\%$, $\frac{5}{8} = \_\_\_\_\_\%$; $\frac{7}{8} = \_\_\_\_\%$

2c) $87\frac{1}{2}\% =$ Fraction , $1/5 = \_\_\_\_\_\%$, 125 % of 60 =

3a) $\frac{1}{2} = \frac{\square}{8}$, $\frac{1}{4} = \frac{\text{⯃}}{8}$ b) Halfway between $\frac{1}{4}$ and $\frac{1}{2}$ is $\frac{\bigcirc}{8}$

4a) $\frac{3}{8} = 3 \times \frac{1}{\square} = 3 \times \_\_\_\_\% = \_\_\_\_\%$

b) $\frac{5}{8} = \_\_\_\_\%$ c) $87\frac{1}{2}\%$ of $40 = \_\_\_\_$.

5) $2\frac{1}{2}$ tenths $= \frac{2\frac{1}{2}}{10} = \frac{\square}{20} = \frac{\bigcirc}{4} = \frac{\text{⯃}}{100} = \_\_\_\_\%$

6) $\frac{2\frac{1}{2}}{10} = 2.5 \div \_\_\_ = .\_\_\_ = \_\_\_\_\%$

7) $37\frac{1}{2}\% = 25\% + \_\_\_\_\% = \frac{1}{4} + \frac{1}{\bigcirc} = \frac{\square}{8}$

8a) 100% of 40 = ____ b) 80% of 40 = $\frac{\bigcirc}{5}$ x 40 = _____ c) 40% of 40 = ____

9a) 180% of 40 = ____ b) 250% of 40 = _____ c) 187.5% of 40 = _____

d) 10% of 50 = ____ e) .10 x 50 = ____ f) 110% of 50 = _____

> Note: Different shapes $\square$ $\bigcirc$ ⯃ *allow* different numbers to be placed in them in the same problem but they do not *require* it.
>
> Same shape in same problem *requires* same number**:**
>
> $\boxed{5} + (5) = 10$ OK
>
> $\boxed{6} + (4) = 10$ OK
>
> $(3) + (7) = 10$ No

SOME NEW, SOME OLD, FRACTIONS, DECIMALS AND PERCENTS | *Unit 27

10a) 40 increased by 10% (of itself) = ___ b) ◯ % of 40 = 44 c) ◯ % of 50 = 60?
Try to do these mentally.

11) Samples for study**:** $\frac{1}{3} = .333\ldots = .33\overline{3} = .\overline{3} = .33\frac{1}{3} = 33\frac{1}{3}\% = 33.\overline{3}\%$

All points . . . are important.

a) **Circle** each equivalent to $\frac{1}{9}$**:**

$.111\ldots$	$.\overline{1}$	$.11\overline{1}$	$11.\overline{1}\%$	$11\frac{1}{9}\%$

b) $\frac{5}{9}$ =._ _ _. . . =.__ = .__ __ $\frac{5}{9}$ = _____ %

c) $\frac{9}{9}$ =._ _ _. . . =.__ = .__ __ $\frac{\ }{9}$ = _____%

A cap above a number or group of numbers indicates that number or group repeats unendingly.
$.\overline{5} = .555\ldots$

Correct to here

12a) Divide 1 by 11 and find repeating pattern. __________ b) $\frac{2}{11}$ = __________

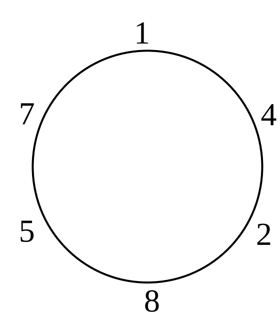
c) $.9090\overline{90}$ = (fraction) _____ *d) Subtract: $\begin{array}{r} .909090\,\overline{90} \\ \underline{-.090909\,\overline{09}} \end{array}$ e) $.\overline{90} - .\overline{09}$ = ___

13a) Divide to find the repeating decimal pattern for $\frac{1}{7}$. ____________
Don't forget the bar over the whole repeating part (called the *period*).

b) If you got part a) correct, the first two digits will be 2 x 7, or 14, the third and fourth will be twice the first two, and the fifth and sixth will be twice the third and fourth, plus 1. Write the repeating decimals for 2, 3, 4, 5, and 6 sevenths. Use patterns as much as you can, even for discovering the first digit. The bar goes over the whole period (repeating part). No division should be necessary.

____

____



14) Put decimals in the blanks.

a) $.13 = .130$ | b) $.07 = .0700$ | c) $.03 = .03000$

$.12\frac{1}{2}$ = ____ | $.06\frac{1}{4}$ = _____ | $.02\frac{1}{8}$ = ______

$.12 = .120$ | $.06 = .0600$ | $.02 = .02000$

You may use your own rule about how to change a percent to a decimal and vice versa. Many people just move the point left or right two places if that's convenient.

SOME NEW, SOME OLD, FRACTIONS, DECIMALS AND PERCENTS *Unit 27

Another approach is to remember that Per**cent** *means* per **hundred:** .38 = $\frac{38}{100}$ = 38%

2.5 = $\frac{250}{100}$ = 250%   293.5 = 293.5 x $\frac{100}{100}$ = $\frac{29350.}{100}$ = 29350%

You can always multiply by $\frac{100}{100}$ because it is a name for 1. But why do it?

Because, num/100 = num percent.

Change to decimals. Almost everything on this page can be done mentally. **NOTE**

Demo 1. $3\frac{1}{8}$% = .03$\frac{1}{8}$ = .03125   **Demo 2.** $8\frac{1}{3}$% = .08$\frac{1}{3}$ = $.08\overline{3}$

15) (To decimals)

a) $12\frac{3}{4}$% = ______ b) $137\frac{1}{2}$% = _____ c) $11\frac{3}{8}$% = _____ d) $142\frac{5}{9}$% = ______

e) .6$\frac{1}{2}$% = _______ f) $\frac{1}{4}$% = ______ g) $12\frac{1}{7}$% = ______ *h) $5\frac{6}{7}$% = _________

16a) $6\frac{1}{4}$% = $\frac{1}{2}$ of _____% . b) $3\frac{1}{8}$% = $\frac{1}{2}$ of _____%. c) $3\frac{1}{8}$% = .03__ __ __

17a) $16\frac{2}{3}$% = $\frac{1}{2}$ of _____%. b) $8\frac{1}{3}$% = $\frac{1}{2}$ of ______ c) $8\frac{1}{3}$% = ____________ (repeating decimal)

d) What is the fraction equivalent of $8\frac{1}{3}$%? ______

e) What is the fraction equivalent of $12\frac{1}{2}$%?______

Ask "Each of these is half of what?"

18a) $\frac{3}{40}$ = $\frac{3}{4} \times \frac{1}{\bigcirc}$ = .__ __ x .___ = .__ __ __ * b) $\frac{3}{80}$ = .375 x .____ =._________

c) $\frac{1}{300}$ = $\frac{1}{3}$ x $\frac{1}{100}$ = $.33\overline{3}$ x .01 = ._______ * d) $\frac{1}{9000}$ = (decimal) ________

e) .0333 . . . = .__ x $.\overline{3}$ = $\frac{1}{10} \times \frac{1}{\square}$ = ____ (fraction) f) $\frac{3}{40}$ = ______ %

To fractions: g) $.\overline{6}$ = ___ h) $.00\overline{6}$ = ______ i) .0875 = _____

What is the **reciprocal** of: j) .002 _____ k) .012$\frac{1}{2}$ _____ l) .00012$\frac{1}{2}$_____ m) $16\frac{2}{3}$% _____

Some New, Some Old, Fractions, Decimals and Percents *Unit 27

Answers

NOTE: Decimal points which appeared in the questions are not repeated here in the answers.

1a) 50, 50, 50 **b)** 25, 25, 25

2a) 8, $12\frac{1}{2}$, $12\frac{1}{2}$, $12\frac{1}{2}$ **b)** $12\frac{1}{2}$, 25, $37\frac{1}{2}$, $62\frac{1}{2}$, $87\frac{1}{2}$ **c)** 7/8, 20, 75

3a) 4, 2 **b)** 3

4a) 8, $12\frac{1}{2}$, $37\frac{1}{2}$ **b)** $62\frac{1}{2}$ **c)** 35

5) 5, 1, 25, 25 **6)** 10, 25, 25 **7)** $12\frac{1}{2}$, 8, 3

8a) 40 **b)** 4, 32 **c)** 16

9a) 72 **b)** 100 **c)** 75 **d)** 5 **e)** 5 **f)** 55

10a) 44 **b)** 110 **c)** 120

11a) All are correct. **b)** .555, $.\overline{5}$, .55, $55\frac{5}{9}$ % **c)** .999, $.\overline{9}$, $.99\frac{9}{9}$, 100%

12a) $.09\overline{0}\overline{9}$ **b)** $.\overline{1}\overline{8}$ **c)** $\frac{10}{11}$ **d)** $.8181\overline{8}\overline{1}$ **e)** $.\overline{8}\overline{1}$

13a) $\frac{1}{7} = .\overline{142857}$

13b) $\frac{2}{7} = .\overline{285714}$, $\frac{3}{7} = .\overline{428571}$, $\frac{4}{7} = .\overline{571428}$, $\frac{5}{7} = .\overline{714285}$, $\frac{6}{7} = .\overline{857142}$

14a) .125 **b)** .0625 **c)** .02125

15a) .1275 **b)** 1.375 **c)** .11375

d) $1.42\overline{5}$ ($1.42\overline{55}$ is equal but less simple.) **e)** .0065 **f)** .0025

g) $.12\overline{142857}$ **h)** $.05\overline{857143}$

16a) $12\frac{1}{2}$ **b)** $6\frac{1}{4}$ **c)** 125

17a) $33\frac{1}{3}$ **b)** $16\frac{2}{3}$ **c)** $.08\overline{3}$ **d)** $\frac{1}{12}$ **e)** $\frac{1}{8}$

18a) 10, 75, 1, 075 **b)** 1 0375 **c)** $.00\overline{3}$ **d)** $.000\overline{1}$ **e)** 1, 3, 1/30

f) 7.5 **g)** $\frac{2}{3}$ **h)** $\frac{1}{150}$ **i)** $\frac{7}{80}$ **j)** 500

k) 80 **l)** 8000 **m)** 6

Not easy

SQUARES AND SHORTCUTS *Unit 28

(No calculator, mostly mental)

0^2 = ___ 0 x any number = 0

1^2 = ___ 1 x any number is that number. So 1 is called the ____________ element.

2^2 = ___ **Fill all boxes and blanks.**

3^2 = ___

4^2 = ___ $4^2 = 2^{\square}$

5^2 = ___ So, the square of any number ending in 5 ends in 5; actually, ends in 25.

6^2 = ___ $6^2 = 4 \times 9 = \_\_\_^{\square} \times \_\_\_^{\square}$

Numbers must be the same in these two exponent boxes.

7^2 = ___

Numbers *may* be different in two blanks.

8^2 = ___ $8^2 = 2^{---} = 4^{---}$

9^2 = ___ $9^2 = 3^2 \times 3^2 = 3^4$

10^2 = ____ $10^2 = 5^2 \times \_\_^2 = 25 \times \_\_$

Use staggered multiplication mentally

11
x11
11
11

11^2 = ____

12^2 = ____ Use staggered addition if you don't remember 1 gross.

13^2 = ____ Staggered addition and *ending*.

How much?

14^2 = ____ Mentally multiply $2^2 \times 7^2$. (4 x 49 is ___less than 4 x 50.) Last sq. in100's

15^2 = ____ Ends in 25. First square in 200's.

Get used to the squares and how to find them mentally.

16^2 = ____

17^2 = ____ Ending. Last one in 200's. It is strangely easy to remember.

18^2 = ____ $2^2 \times 9^2 = 4 \times 81 =$ ____

19^2 = ____ Ends in 1, no other nearby square does. } Probably should be memorized

20^2 = ____

21^2 = ____ Staggered addition and ending

22^2 = ____ Staggered addition and ending

23^2 = ____ Staggered addition and ending (not easy)

24^2 = ____ $12^2 \times 2^2 = 144 \times 4 = 288 \times \_\_ =$ _____ (Not really a big help)

25^2 = ____ Ends in 25. First square in 600's. $25^2 = 5^{\square}$

SQUARES AND SHORTCUTS *Unit 28

To get a third column number, say 5, go to the number to its left, 9, and subtract the number above it, 4. 9 – 4 = 5. The next number is 16 – 9 = 7.

$0^2 = 0$		NOTES
$1^2 = 1$	1	
$2^2 = 4$	3	
$3^2 = 9$	5	
$4^2 = 16$	___	$4^2 = 2^4$
$5^2 = 25$	___	
$6^2 = 36$	___	$6^2 = 4 \times 9 = \underline{2^2} \times \underline{3^2}$
$7^2 = 49$	___	
$8^2 = 64$	___	$8^2 = 2^6 = 4^3$
$9^2 = 81$	___	$9^2 = 3^4$
$10^2 = 100$	___	$10^2 = \underline{5^2} \times \underline{2^2} = 25 \times 4$
$11^2 = 121$	___	Note the squares as you go along in the exercises.
$12^2 = 144$	___	
$13^2 = 169$	___	
$14^2 = 196$	___	$7^2 \times 2^2 = 49 \times 4 =$ 4 less than $50 \times 4 = 196$
$15^2 = 225$	___	Ends in 25. First in 200's
$16^2 = 256$	___	
$17^2 = 289$	___	
$18^2 = 324$	___	$2^2 \times 9^2 = 4 \times 81$
$19^2 = 361$	___	Last in the 300's
$20^2 = 400$	___	
$21^2 = 441$	___	
$22^2 = 484$	___	
$23^2 = 529$	___	(Not easy)
$24^2 = 576$	___	(288 x 2 - not easy)
$25^2 = 625$	___	$25^2 = 5^4$

1) Notice in the squares column (your answers) how quickly the squares get larger. From 2^2 to 3^2 is 5, but from 10^2 to 11^2 is 21. From 20^2 to 21^2 is ______.

2) Without using a calculator, find how far it is from 1000^2 to 1001^2. Do 1000^2 mentally. Work out 1001^2 and notice the interesting multiplication. Could it be done mentally? Difference = ______.

3) Now finish the differences between all the consecutive squares to the bottom if you haven't already.

4) No doubt you saw the set of odd numbers emerge from this. What would be the difference from 25^2 to 26^2? ______ So, mentally, you know $26^2 =$ ___

5) Here is an activity called "What's the rule?" Examples are given and you show that you know the rule by solving more exercises.

		Another rule:
3 ⌡ 9		
7 ⌡ 49		3 ⌠ 5
17⌡ $\boxed{}$		12 ⌠ 23
30⌡ $\boxed{}$		25 ⌠ 49
$\boxed{}$ ⌡ 196		20 ⌠ $\boxed{}$
$\boxed{}$ ⌡ 10,000		$\boxed{}$ ⌠ 99

6) 7⇐ will mean add the first 7 odd numbers. One rule after another:

7 ⇐ $\boxed{}$	
8 ⇐ $\boxed{}$	3 ⇐ ⌠ 17
9 ⇐ $\boxed{}$	5 ⌠ ⇐ 81
$\boxed{}$ ⇐ 121	5 ⇐ ⌠ $\boxed{}$
$\boxed{}$ ⇐ 625	(5 ⌠) ⌠ $\boxed{}$

Squares and Shortcuts *Unit 28

Answers 1 – 6

$0^2 = 0$

$1^2 = 1$ 1   1 is the <u>identity</u> for mult.

$2^2 = 4$ 3

$3^3 = 9$ 5

$4^2 = 16$ 7 $4^2 = 2^4$

$5^2 = 25$ 9

$6^2 = 36$ 11 $6^2 = 4 \times 9 = \underline{2^2} \times \underline{3^2}$

$7^2 = 49$ 13

$8^2 = 64$ 15 $8^2 = 2^6 = 4^{\underline{3}}$

$9^2 = 81$ 17 $9^2 = 3^4$

$10^2 = 100$ 19 $10^2 = 5^2 \times \underline{2}^2 = 25 \times \underline{4}$

$11^2 = 121$ 21

$12^2 = 144$ 23

$13^2 = 169$ 25

Get used to the squares.

$14^2 = 196$ 27   (4 x 49 is <u>4</u> less)

$15^2 = 225$ 29   Ends in 25.First in 200's

$16^2 = 256$ 31   ($16^2 = 4^4 = 2^8$)

$17^2 = 289$ 33

$18^2 = 324$ 35 $2^2 \times 9^2 = 4 \times 81 = \underline{324}$

$19^2 = 361$ 37

$20^2 = 400$ 39

$21^2 = 441$ 41

$22^2 = 484$ 43

$23^2 = 529$ 45   (Not easy)

$24^2 = 576$ 47   288 x <u>2</u> = <u>576</u> (not easy)

$25^2 = 625$ 49 $25^2 = 5^4$

1) Notice in the squares column how quickly the squares get larger. From 2^2 to 3^2 is 5, but from 10^2 to 11^2 is 21. From 20^2 to 21^2 is <u>41</u>

2) Without using a calculator, find how far it is from 1000^2 to 1001^2. Do 1000^2 mentally. Work out 1001^2 and notice the interesting multiplication. Could it be done mentally? Difference is <u>2001</u>.

3) Now finish the differences between all the consecutive squares in the columns to the left. Finish that column from 1 to the bottom.

4) No doubt you saw the set of odd numbers emerge from this. What would be the difference from 25^2 to 26^2? <u>51</u> So, mentally, you know 26^2 = <u>676</u>

5) Here is an activity called "What's the rule?" Examples are given and you show you know the rule by solving more exercises. Another:

3 ⌡ 9	
7 ⌡ 49	3 ⌠ 5
17 ⌡ <u>289</u>	12 ⌠ 23
30 ⌡ <u>900</u>	25 ⌠ 49
<u>14</u> ⌡ 196	20 ⌠ <u>39</u>
<u>100</u> ⌡ 10,000	<u>50</u> ⌠ 99

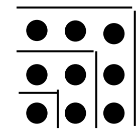
6) 7⇐ will mean add the first 7 odd numbers. The first 4 odds 1,3,5,7⇐16. One rule after another:

7⇐ <u>49</u>	3⇐ ⌠ 17
8⇐ <u>64</u>	5 ⌠ ⇐ 81
9⇐ <u>81</u>	5 ⇐ ⌠ <u>49</u>
<u>11</u>⇐ 121	(5 ⌠) ⌠ <u>17</u>
<u>25</u>⇐ 625	

SQUARES AND SHORTCUTS *Unit 28

On the previous page you probably saw that ⌡ and ⇐ gave the same result, even though one rule is for squaring and the other for adding odds. Why is that??

The diagram to the right gives a picture of what is happening with the numbers.



8) Extend the figure up and to the right one more column and row. This should give a new addition of 7 dots, making a square with 16 dots. Use scrap paper if you wish.

9) The next such addition will give ___ new dots, making a square with a total of ___ dots.

10) Mentally, or use scrap paper continue until the square contains 81 dots. How many dots has the last addition?_____

11) □ ⌡ 81; □ ⇐ 81; □ ⇐ ⇐ 81

12) 7 x 7 =
 8 x 6 = ___

13) 6 x 6 = ___
 5 x 7 = ___

14) 12^2 = ___
 11 x 13 = ___

15) 40 x 40 = _____
 41 x 39 = _____

16) 29 x 31 = _____
 21 x 19 = _____

17) 200^2 = ________
 40,000 - 1 = ________
 199 x 201 = ________

18) 19 x 21 = ______
 19 x 210 = ______
 190 x 21 = _______
 190 x 210 = _______

19) 14 x 16 = _____
 24 x 26 = _____
 18^2 = _____
 17 x 19 = _____

Try not to use the table of squares.

20) 15 x 17 = ______
 150 x _____ = 255,000

21) 160 x 1800 = __________

22) 1200 x _______= 16,800

23) 23 x 2500 = ________

Watch the patterns:

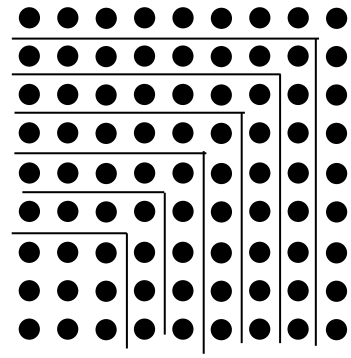
24) 10^2 = ____
 9 x 11 = ___
 8 x 12 = ___
 7 x 13 = ___ (Be sure it's right.)
 6 x 12 = ___

25) Did you really see the pattern in example 24 ? Let's find out:

 14^2 = ____
 13 x 15 = ____ (1 less x 1 more)
 12 x 16 = ____ (2 less x 2 more)
 11 x 17 = ____ (3 less x 3 more)
 10 x 18 = ____ (4 less x 4 more)
 ___x___= ____ (____________)

SQUARES AND SHORTCUTS *Unit 28

Answers



8) Extend the figure up and to the right one more column and row. This should give a new addition of 7 dots, making a square with 16 dots.

9) The next such addition will give 9 new dots, making a square with a total of 25 dots.

10) Neatly continue until the square contains 81 dots. How many dots has the most recent addition? 17

11) 9 ⌡ 81; 9 ⇐ 81; 3 ⇐ ⇐ 81

12) 7 x 7 = 49
8 x 6 = 48

13) 6 x 6 = 36
5 x 7 = 35

14) 12^2 = 144
11 x 13 = 143

15) 40 x 40 = 1600
41 x 39 = 1599

16) 29 x 31 = 899
21 x 19 = 399

17) 200^2 = 40,000
40,000 - 1 = 39,999
199 x 201 = 39,999

18) 19 x 21 = 399
19 x210 = 3990
190 x 21 = 3990
190 x 210 = 39,900

19) 14 x 16 = 224
24 x 26 = 624
18^2 = 324
17 x 19 = 323

20) 15 x 17 = 255
150 x 1700 = 255,000

21) 160 x 1800 = 288,000

22) 1200 x 14 = 16,800

23) 23 x 2500 = 57,500

Watch the patterns:

24) 10^2 = 100
9 x 11 = 99
8 x 12 = 96
7 x 13 = 91 (Be sure it's right.)
6 x 12 = 72

25) Did you really see the pattern in exercise 24? Let's find out:
14^2 = 196
13 x 15 = 195 (1 less x 1 more)
12 x 16 = 192 (2 less x 2 more)
11 x 17 = 187 (3 less x 3 more)
10 x 18 = 180 (4 less x 4 more)
9 x 19 = 171 (5 less x 5 more)

Mental Arithmetic 1 — Unit 29

No Calculator

1) 10 x 162 = ______
2) 5 x 162 = ______
3) 20 x 162 = ______
4) 10 x 84 = ______
5) 5 x 84 = ______
6) 20 x 84 = ______
7) 5 x 624 = ______
8) 20 x 624 = ______
9) 5 x 468 = ______
10) 20 x 321 = ______
11) ¼ of 24 = ______
12) ¾ of 24 = ______
13) 1/6 of 18 = ______
14) 5/6 of 18 = ______
15) 10 x 844 = ______
16) 5 x 844 = ______
17) 20 x 844 = ______
18) 5 x 1426 = ______
19) 20 x 1321= ______
20) 50 x 1426= ______
21) 1/8 of 40 = ______
22) 3/8 of 40 = ______
23) 7/8 of 40 = ______
24) 4/5 0f 30 = ______
25) 100 + 124 =______
26) 99 + 124 = ______
27) 299 + 24 = ______
28)199 + 124= ______

29) 199 + 375 = ______
30) 299 + 417 = ______
31) 628 + 198 = ______
32) 29 ½ + 1 ½ =______
33) 60 - 2 = ______
34) 60 - 2 1/3 = ______
35) 60 - 1/3 = ______
36) 60 - 2 2/3 = ______
37) 40 - 4 2/3 = ______
38) 6 x 6 = ______
39) 12 x 6 = ______
40) 12 x 12 = ______
41) 24 x 12 = ______
42) 8 x 8 = ______
43) 8 x 16 = ______
44) 16 x 16 = ______
45) 7 x 8 = ______
46) 14 x 8 = ______
47) 14 x 16 = ______
48) 7 x 16 = ______
49) 28 x 16 = ______
50) 21 x 16 = ______
51) 20 - 2 1/3 = ______
52) 50 - 11/5 = ______
53) 50% of 20 = ______
54) 25% of 20 = ______
55) 12 ½% of 20=______
56) 25% of 32 = ______

57) 12 ½% of 40 = ______
58) 10 x 35 = ______
59) 9 x 35 = ______
60) 11 x 35 = ______
61) 20 - 3/7 = ______
62) 50 - 2 2/7 = ______
(Not 48 5/7)
63) 11 x 36 = ______
64) 11 x 27 = ______
67) 632 - 100 = ______
68) 632 - 99 = ______
69) 545 - 95 = ______
70) ½ of $16 = ______

Some of these are Not Fair
Answer NF or give exact answer
No calculator remember

71) $\sqrt{9}$ = _____
72) $\sqrt{90}$ = ______ (not 30)
73) $\sqrt{900}$ = ______
74) $\sqrt{9000}$ = ______
75) $\sqrt{90000}$ = _______
76) $\sqrt{900{,}000}$ = ________
77) $\sqrt{9{,}000{,}000}$ = ________
78) $\sqrt{9 \times 10^8}$ = _______
*79) $\sqrt{2^6 \times 10^{12}}$ = _________
80) 17^2 = ______
81) 16 x 18 = ______
82) 19 x 15 = ______

MENTAL ARITHMETIC 1 Unit 29

Answers

1) 10 x 162 = 1620
2) 5 x 162 = 810
3) 20 x 162 = 3240
4) 10 x 84 = 840
5) 5 x 84 = 420
6) 20 x 84 = 1680
7) 5 x 624 = 3120
8) 20 x 624 = 12480
9) 5 x 468 = 2340
10) 20 x 321 = 6420
11) ¼ of 24 = 6
12) 3/4 of 24 = 18
13) 1/6 of 18 = 3
14) 5/6 of 18 = 15
15) 10 x 844 = 8440
16) 5 x 844 = 4220
17) 20 x 844 = 16,880
18) 5 x 1426 = 7130
19) 20 x 1321= 26,420
20) 50 x 1426= 71,300
21) 1/8 of 40 = 5
22) 3/8 of 40 = 15
23) 7/8 of 40 = 35
24) 4/5 of 30 = 24
25) 100 + 124=224
26) 99 + 124 =223
27) 299 + 24 = 323
28) 199 + 124=323

29) 199 + 375 = 574
30) 299 + 417 = 716
31) 628 + 198 = 826
32) 29 ½ + 1 ½ = 31
33) 60 - 2 = 58
34) 60 - 2 1/3 = 57 2/3
35) 60 - 1/3 = 59 2/3
36) 60 - 2 2/3 = 57 1/3
37) 40 - 4 2/3 = 35 1/3
38) 6 x 6 = 36
39) 12 x 6 = 72
40) 12 x 12 = 144
41) 24 x 12 = 288
42) 8 x 8 = 64
43) 8 x 16 = 128
44) 16 x 16 = 256
45) 7 x 8 = 56
46) 14 x 8 = 112
47) 14 x 16 = 224
48) 7 x 16 = 112
49) 28 x 16 = 448
50) 21 x 16 = 336
51) 20 - 2 1/3 = 17 2/3
52) 50 - 11/5 = 47 4/5
53) 50% of 20 = 10
54) 25% of 20 = 5
55) 12 1/2% of 20 = 2 1/2
56) 25% of 32 = 8

57) 12 ½ % of 40 = 5
58) 10 x 35 = 350
59) 9 x 35 = 315
60) 11 x 35 = 385
61) 20 - 3/7 = 19 4/7
62) 50 - 2 2/7 = 47 5/7
(Not 48 5/7)
63) 11 x 36 = 396
64) 11 x 27 = 297
67) 632 - 100 = 532
68) 632 - 99 = 533
69) 545 - 95 = 450
70) ½ of \$16 = \$8

71) $\sqrt{9} = 3$
72) $\sqrt{90}$ = NF
73) $\sqrt{900} = 30$
74) $\sqrt{9000}$ = NF
75) $\sqrt{90000} = 300$
76) $\sqrt{900{,}000}$ = NF
77) $\sqrt{9{,}000{,}000} = 3{,}000$
78) $\sqrt{9x10^8} = 3x10^4$ or 30000
79) $\sqrt{2^6 x10^{12}} = 2^3 x10^6 = 8{,}000{,}000$
80) $17^2 = 289$
81) 16 x 18 = 288
82) 19 x 15 = 285

MENTAL ARITHMETIC 2

Unit 30

Many of the ideas in "Mental Arithmetic" may already be known to you. That's okay. Do the exercises anyway. "Mental Arithmetic 1", "2" and "Squares and Shortcuts" are all worthwhile but mathematical "tastes" differ. Keep an open mind and try them out as opportunities come up.

1) 1/2 of $16 = ___ 2) 1/2 of 16/17 = ____ 3) 1/4 of 16/17 = ____

4) 199 + 724 = ____ 5) 724 + 198 = ____ 6) 1/3 of 18/19 = ____

7) 1/4 of 48/49 = _____ 8) 556 + 197 = ____ 9) 2/3 of 6/11 = _____

10) 2 1/2 x 20 means multiply both 2 x 20 and 1/2 x 20:

2 x 20 = ___, 1/2 x 20 (or 1/2 *of* 20) = ____, 2 1/2 x 20 = ____

11) 3 1/2 x 20 = (3 x 20) + (1/2 x 20) = ____

12) 3 1/2 x 50 = ____ 13) 3 1/3 x 30 = ____ 14) 2 1/5 x 20 = ____

Check answers to 1 - 14 before going on)

15) 19 x 21 = ____ 16) 19 1/7 x 21 = ____ 17) 21 x 2/7 = ____

*18) 19 2/7 x 21 = ____ 19) 3 3/7 x 21 = ____ 20) 3 1/4 x 32 = ____

21) 300 x 40 = _____ 22) 50 x 400 = _____ 23) ____ x 80 = 400

<u>Double-half:</u> Sometimes it's easier to multiply two numbers if you first double one and half the other:

24) 3 x 18 = 6 x 9 = ____ 25) 4 1/2 x 14 = 9 x ___ =___ 26) 14 x 50 = 7 x ___ = ____

27) 18 x 50 = _____ 28) 8 x 15 = ________ 29) 18 x 400 = 9 x ___ = ___

30) 40 x 40 = ____ 31) 160 x 300 = _______ 32) 1 1/2 x 22 = _____

33) 3 1/2 x 14 = _____ 34) 180 x 35 = _________ 35) 1600 x 2500 = ________

36) 1200 x 15 = _______ 37) 180 x 500 = ________ 38) 16 x 50 = _____

39) 400 x 35 = _______ 40) 1800 x 40 = ________ 41) 1400 x 25 = _________

Miscellaneous Mentals:

42) 29 x 31 = _____ 43) 102 x 98 = __________ 44) 15^2 = _____

45) 13 x 17 = ______ 46) 12 x 18 = ___________ 47) 298 + 572 = _________

48) 5 x $4.99 = _______ 49) 12 x $2.98 = _______ 50) 500 x 432 = _________

*51) 149 X 151 = _______ 52) 13^2 = ____________ 53) 140 x 1600 = _________

*54) 130 x 170 x 10^4 = ______________

MENTAL ARITHMETIC 2 | Unit 30

Answers

1) 1/2 of $16 = $8 **2)** 1/2 of 16/17 = 8/17 **3)** 1/4 of 16/17 = 4/17

4) 199 + 724 = 923 **5)** 724 + 198 = 922 **6)** 1/3 of 18/19 = 6/19

7) 1/4 of 48/49 = 12/49 **8)** 556 + 197 = 753 **9)** 2/3 of 6/11 = 4/11

10) 2 1/2 x 20 means multiply both 2 x 20 and 1/2 x 20:

2 x 20 = 40 , 1/2 x 20 (or 1/2 *of* 20) = 10, 2 1/2 x 20 = 50

11) 3 1/2 x 20 = (3 x 20) + (1/2 x 20) = 70

12) 3 1/2 x 50 = 175 **13)** 3 1/3 x 30 = 100 **14)** 2 1/5 x 20 = 44

----

15) 19 x 21 = 399 **16)** 19 1/7 x 21 = 402 **17)** 21 x 2/7 = 6

18) 19 2/7 x 21 = 405 **19)** 3 3/7 x 21 = 72 **20)** 3 1/4 x 32 = 104

21) 300 x 40 =12000 **22)** 50 x 400 = 20,000 **23)** 5 x 80 = 400

----

24) 3 x 18 = 6 x 9 = 54 **25)** 4 1/2 x 14 = 9 x 7 = 63 **26)** 14 x 50 = 7 x 100 = 700

27) 18 x 50 = 900 **28)** 8 x 15 = 120 **29)** 18 x 400 = 9 x 800 = 7200

30) 140 x 40 = 5600 **31)** 160 x 300 = 48,000 **32)** 1 1/2 x 22 = 33

----

33) 3 1/2 x 14 = 49 ***34)** 180 x 35 = 6300 **35)** 1600 x 2500 = 4,000,000

36) 1200 x 15 = 18,000 **37)** 180 x 500 = 90,000 **38)** 16 x 50 = 800

39) 400 x 35 = 14,000 **40)** 1800 x 40 = 72,000 **41)** 1400 x 25 = 35,000

----

42) 29 x 31 = 899 **43)** 102 x 98 = 9,996 **44)** 15^2 = 225

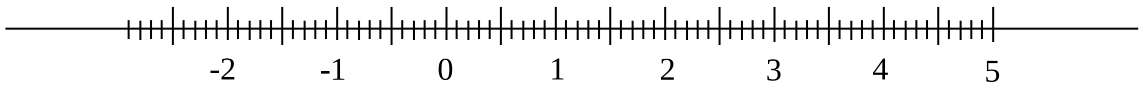
45) 13 x 17 = 221 **46)** 12 x 18 = 216 **47)** 298 + 572 = 870

48) 5 x $4.99 = $24.95 **49)** 12 x $2.98 =$35.76 **50)** 500 x 432 = 216,000

***51)** 149 X 151 = 22,499 **52)** 13^2 = 169 **53)** 140 x 1600 = 224,000

***54)** 130 x 170 x 10^4 = 221,000,000 or 221 x 10^6

RECIPROCALS ON THE NUMBER LINE *Unit 31



Dealing with reciprocals on the number line brings up some interesting ideas about infinity, but before we can deal with these, we have a loose end to pick up.

We worked with positive and negative numbers in "Number Line Jumps" but we avoided problems like -5 x (-3) = ? The number line does not provide a good "model" for multiplying negatives. We will have to use persuasive argument, painful as that may be.

3 x 5 = 15	No arguing with this.
3 x (-5) = -15	Easily seen on the number line as 3 jumps of 5 in the *negative* direction.
-5 x 3 = -15	Must be the same as 3 x (-5) if multiplication is <u>commutative</u>. (same both ways.)
-5 x -3 = 15	If -15 were correct, then **-5 x 3 = -15** (shown just above) ***<u>and</u>*** **-5 x (-3) = -15** would <u>both</u> be true. Not acceptable. -3 ≠ 3.

<u>Consequences</u>

If 3 x 5 = 15 then 15 ÷ 5 = 3 and 15 ÷ 3=5.

If 3 x (-5) = -15 then -15 ÷ (-5) = 3 and (-15) ÷3 = -5.

If -5 x 3 = -15 then -15 ÷ 3 = -5 and -15 ÷ (-5) = 3.

If -5 x (-3) = 15 then 15 ÷ (-3) = -5 and 15 ÷ (-5) = -3.

For ease of remembering, the content of the boxes below follow a different order than above.

<u>x Summary</u>

(+) x (+) = +
(-) x (-) = +
(+) x (-) = -
(-) x (+) = -

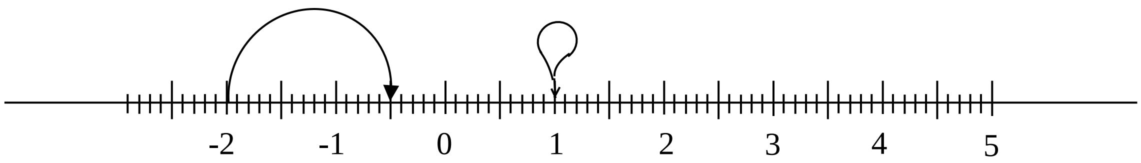
<u>÷ Summary</u>

(+) ÷ (+) =+
(-) ÷ (-) = +
(+) ÷ (-) = -
(-) ÷ (+) = -

Multiplying or dividing two like-signed numbers gives positive answer.

Multiplying or dividing two unlike-signed numbers gives negative answer.

Reciprocals on the Number Line *Unit 31



1) This number line above is divided into tenths. Each of the numbers -2 and 1 has an arrow drawn from itself to its reciprocal. The reciprocal of -2 is (be careful) ____ and the reciprocal of 1 is ____.

2) As shown for exercise 1, <u>draw 5 arrows,</u> one from each of the points -1, .2, .3, .4, .5 to its reciprocal and record each reciprocal in blanks a) through e):

*a) -1 ____ b) .2 ____ c) .3 _____ (Estimate on no. line) d) .4 ____ e) .5 _____

3) Above the arched arrows that you drew in Exercise 2, draw another arrow which appears to leave the paper at the right-hand edge, but leave some room above it. Label the arrow "to 10". Where did it start? _____

4) Draw a somewhat similar arrow which <u>comes</u> <u>onto</u> the paper above the "to 10" arrow. Label it "from 50" and squeeze it down to its approximate landing place which is ______

***Pause here* and correct exercises 1-4. Answers are on page 3 of this unit.**

5) Any positive number less than 1 has a reciprocal greater than _____.

*6) Any <u>negative</u> number <u>greater</u> than -1 has a reciprocal ______________ (greater/less) than -1.

7) The distance between 1.25 and its reciprocal is __________ (decimal answer).

*8) In what interval lie <u>all the reciprocals</u> of the members of this set**:** all positive numbers greater than 1? Between ______ and ______.

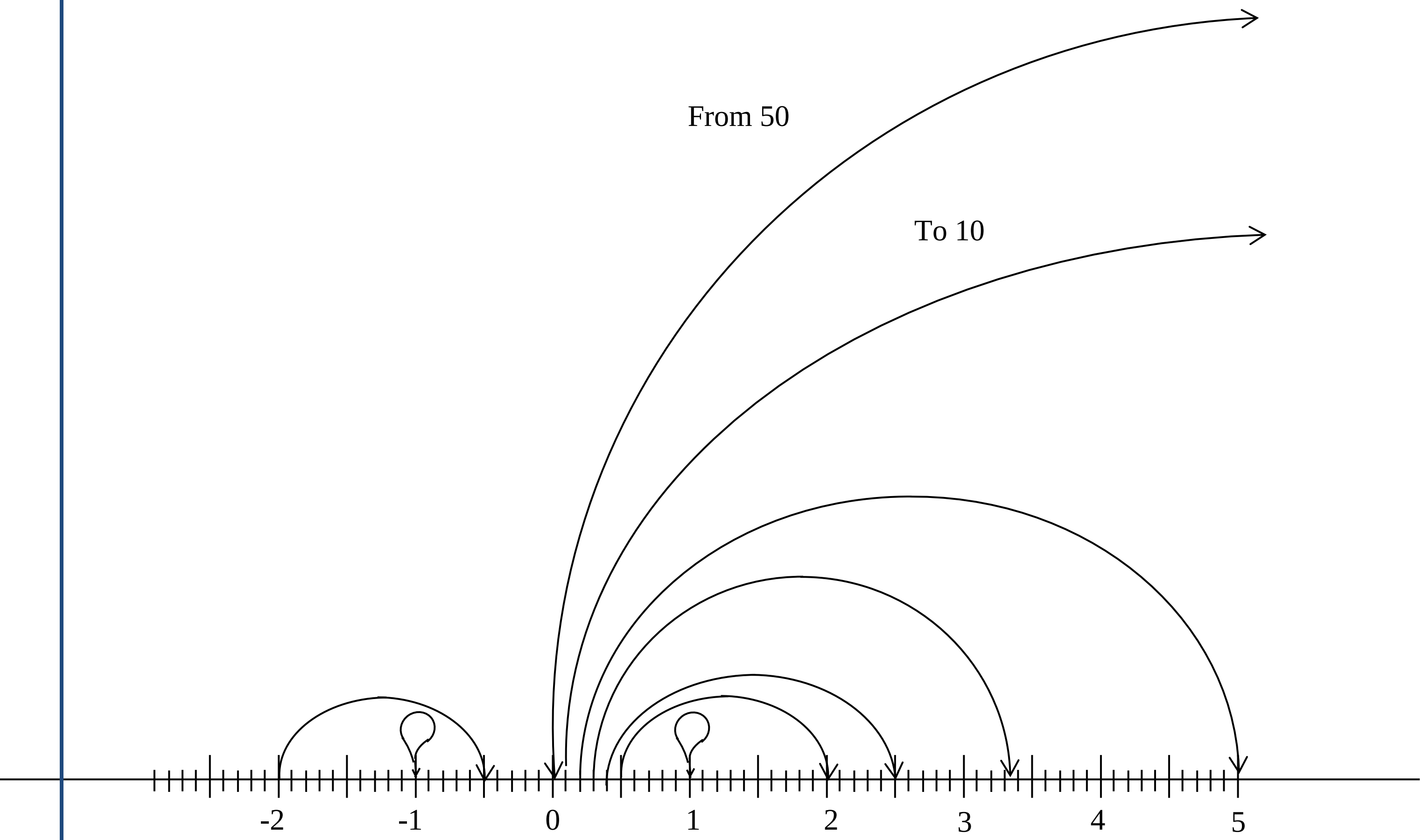
Reciprocals on the Number Line

***Unit 31**

Answers 1 – 4

1) $-\frac{1}{2}$.,...1. **2a)** -1 **b)** 5 **c)** $3\frac{1}{3}$ or $3.\overline{3}$ **d)** $2\frac{1}{2}$ **e)** 2

<u>Please Note</u>: When we say "greater than one" we are excluding 1. When we say "between 1.25 and its reciprocal", we do not say whether we are excluding 1.25 and its reciprocal. But this <u>does not</u> affect the answer to exercise 7. *A point does not occupy any space*, so the length of a "piece of a line" (a segment) remains the same whether its endpoints are included or excluded.



3) .1 **4)** 1/50 or $\frac{1}{50}$ **Resume with exercise 5, page 2 of this unit.**

RECIPROCALS ON THE NUMBER LINE *Unit 31

(No calculator)

9) In each of the following, tell the distance between the given number and its reciprocal. Give exact answer in fraction or mixed number form.

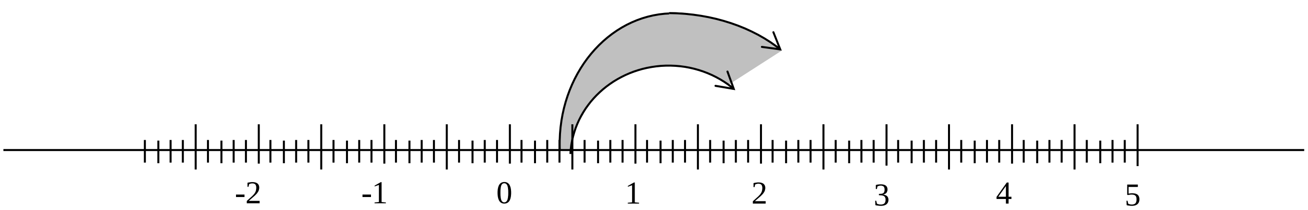
a) 1/2 _____ b) 5 _____ c) 7 _____

d) 20 _______ e) 100 _________ f) 10,000__________ (Not 2)

10) Look at the positive side on your number line drawing (or its answer version on the previous page).

a) If the distance is large between a number and its reciprocal, one of them is near ___.

b) If the distance is small between a number and its reciprocal, each of them is near ___.

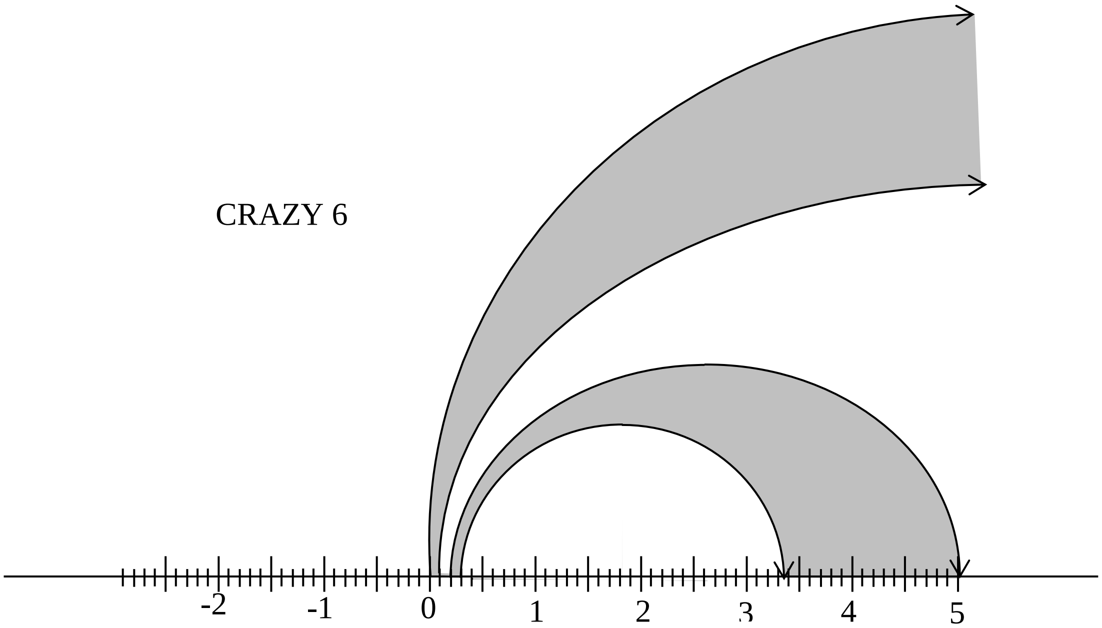


11) On the number line above we are letting the shading represent arrows from *every point* between .4 and .5 on their ways to their respective reciprocals. Give the interval of the reciprocals at the destinations and complete the drawing.

Interval: _____ to ______ .

RECIPROCALS ON THE NUMBER LINE *Unit 31

Notice that the reduced and shaded Crazy 6 below will contain the answer to exercise 11 under its lowest level of shading.



12) For all the numbers from 1/2 to 1, in what interval do their reciprocals fall? This answer is not a shaded area. ___________

13) Consider the interval between 0 and 1, not including either. Describe as best you can the interval of all the reciprocals of the numbers in that 0 to 1 interval. ______________

____

____

This is truly an important idea which brings up some related ideas. There is no need to write responses to the questions or comments A-F. They are there to think about and most will have responses in the next 2 or 3 units. Study A – F now.

A. Every number between 0 and 1 has a reciprocal greater than 1.

B. How many numbers of any kind are there between 0 and 1 anyway? $\aleph_0$? You remember $\aleph_0$, Aleph Null, the number of counting numbers, from Unit 12, A Glimpse into the Infinite. $\aleph_0$ is an infinite number which Georg Cantor called a transfinite number. He said that $\aleph_0$

RECIPROCALS ON THE NUMBER LINE *Unit 31

was the *first* or the *lowest* transfinite number. We said earlier that this suggests there are larger infinities than the counting numbers {1, 2, 3, 4, . . .}.

C. Might the number of *all* the positive numbers including fractions (and their decimal equivalents) be the next infinity?

D. Do such fractions together with the counting numbers "fill up" the number line?

E. Are there other kinds of numbers on the number line? What could they be?

F. Do you think there are many more numbers between 0 and 1 than between 0 and 1/50?

Answers 5 – 13

5) 1

6) Less than. If your answer was “greater than”, think about why “less than” is correct.

7) 0.45 **8)** Between 0 and 1

9a) 1½ **b)** 4 4/5 **c)** 6 6/7 d) 19 19/20 **e)** 99 99/100 **f)** 9999 9999/10000

10a) 0 **b)** 1 **11)** 2 to 2.5 **12)** Between 1 and 2

13) “From 1 (not inclusive) to infinitely large", but this does not designate any particular infinitely large number. Some books would say "From 1 to infinity" (???) But this seems not very informative, either, without further definition.

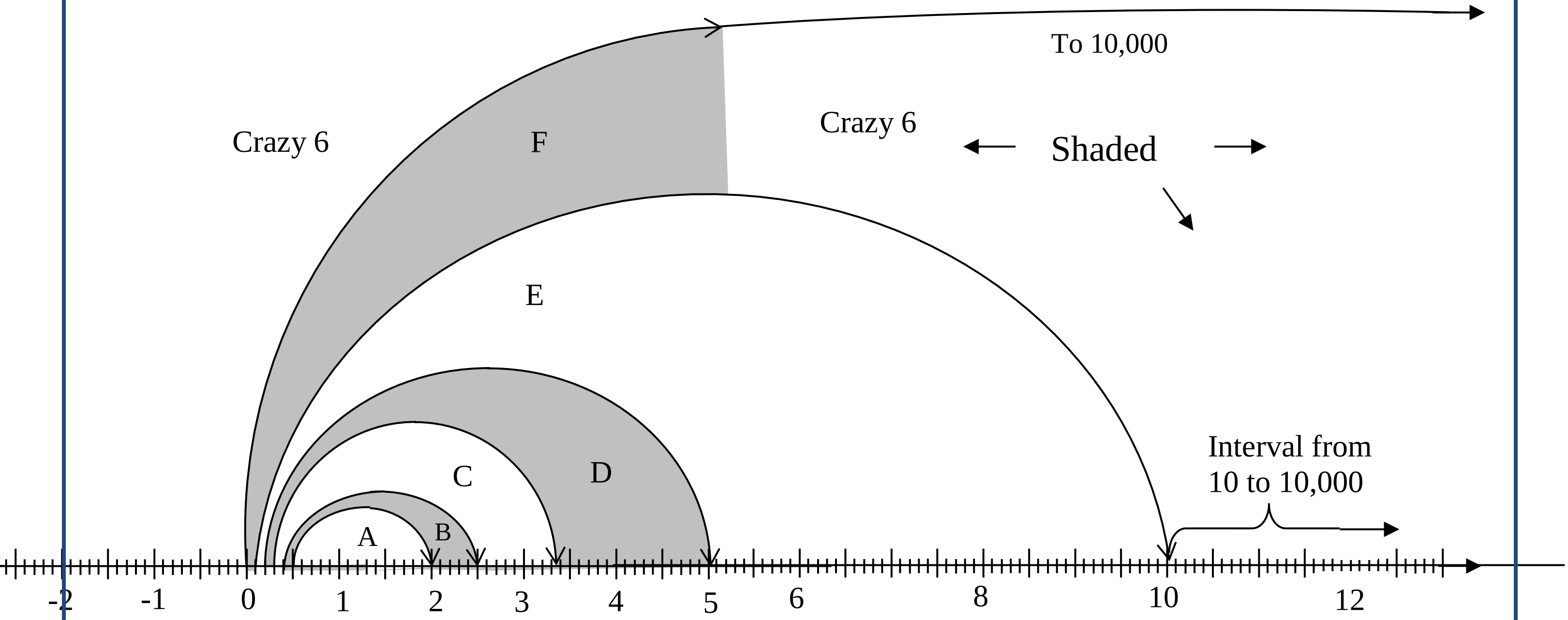
Your opinion**:** Pretend that the ideas of A - F have not yet been determined by anyone. Do you think that A - F represent questions to be answered by discovery, like finding gold? Or, are all these things "made up" *by people*, using previous mental constructions (like counting, fractions, number line, infinity) and expanding them.

____

____

There is no "right or wrong" answer. It is only opinion.

A BIGGER INFINITE? **Unit 32



Recall that a tiny change toward zero in the left interval of Area F produces a huge change in the right interval. In fact, we could start so close to zero in the left interval, that the right interval would be *millions* of *miles* long, still with all numbers in the gigantic right interval having *all* their reciprocals in the tiny left interval!!

It is time now to investigate further just what kinds of numbers are on the number line that would permit such astonishing results. For convenience, we will focus only on zero and the positive numbers. Zero is neither positive nor negative.

> "Zero is like the present**,** which separates the past from the future, but is a part of neither." -- Anonymous

Also we know that 0 x $\boxed{}$ = 1 has <u>no solution</u>; so 0 has no reciprocal.

An infinite set can be put into 1 - 1 correspondence with a <u>part</u> of itself (with a <u>proper</u> subset of itself). We saw this in Unit 12, "A glimpse into the infinite". We saw a one-to-one matching between the set of even numbers, a *part* of the set of natural numbers, and the natural numbers themselves. This was an astounding idea to many mathematicians, even though Galileo had used the idea earlier.

A BIGGER INFINITE? **Unit 32

Georg Cantor (Remember Georg? The pronunciation is with hard G's, as in gag, but sounds like ghee-yorg.) decided to find how many fractions there really are on the number line. Here we mean by fraction a natural number divided by a natural number. These are called *rational numbers* because each is, or can be *shown* to be, the *ratio* of one natural number to another (like .5 = 1/2). Cantor decided to see if they were "**countable**"**;** that is, to see if the rational numbers could be put into one-to-one correspondence with the counting, or natural numbers.

Most people would not even think of doing such a thing. Who would need to assume or reject what was obvious**:** "Of course there are more fractions than counting numbers**.** Look at the number line and see all the fractions between any pair of whole numbers."

But Georg Cantor was a true mathematical pioneer. Starting in the year 1871, he created the Theory of Sets. This included a theory of infinity which, among other sources, is described in "The World of Mathematics" by James R. Newman, Vol. 3, starting on page 1593, and is in the book list at the end of these units. A storm of controversy arose from his efforts and only increased as he enlarged his work. It has not quite died even now.

Cantor knew well that *infinite* cardinal numbers such as $\aleph_0$ do not behave like the familiar cardinal numbers. Cardinal numbers, like 127 or 14,672, answer the question "How many?" as opposed to ordinals like 3rd or 36th which answer the question "Which one?" Having satisfied himself about the need to "count" the rationals, he then needed a way to *arrange* the fractions so they are seen clearly to be in one-to-one-correspondence with the natural numbers. How does one arrange all the fractions in *any* particular order?

We should point out here that the next diagram represents an immense achievement. After a little study the structure will be clear and simple. But what Cantor did yet again was to invent a way to use one to one correspondence to show an astounding result:

The fractions with natural numbers for numerator and denominator are *countable*! They have the same cardinal number as the counting numbers: $\aleph_0$**.** Cantor's scheme is shown on the following page.

A BIGGER INFINITE? **Unit 32

Cantor decided upon the following:

$$\begin{array}{cccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & . & . & . \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & . & . & . \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & . & . & . \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & . & . & . \\
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & .
\end{array}$$

This is the arrangement of the fractions that permits one-to-one correspondence to be shown. The top *row* has every numerator as 1; the first *column* has every *denominator* as 1.

No one could *show* the entire 1-to-1 matching of two infinite sets such as the rational numbers and the natural numbers. What he shows is a plan that illustrates convincingly that a 1-to-1 matching actually does exist, even though we cannot write all of it. Just start counting at 1/1 and follow the arrows. Duplicates are discussed below.

Cantor had already faced strong opposition for his work with $\aleph_0$ as the cardinal number of the natural numbers and of infinite subsets of those numbers such as the odds and evens. $\aleph_0$ was assigned to each of those subsets, also.

Study the array (rows and columns) of numbers to see if all the rational numbers are going to be picked up by following the arrows. 3/5 is in row 3, column 5.

1) 11/20 will be in row ____, column ____.
2) 20/11 will be in row _____, column _____.
3) The fraction in row 100, column 17 is ______.
4) **Continue** the array (rows and columns) for 3 more columns and rows, following the dots to the right and below. Your final diagonal entry should be 8/8.
5) Perhaps you have detected that we are going to count duplicate numbers. Maybe. Give as an example a pair of duplicates: ______________

Notice that the sum of the numerator and denominator is the same for every fraction in each diagonal set of arrows.

Correct Exercises 1 – 5 now. The answers are on page 5 of this unit.

A BIGGER INFINITE? **Unit 32

The following value for 2^{800} was multiplied out by the author's oldest computer. This is close to the biggest number the 40 year old Kaypro computer can handle.

2^{800} = 6668014432879854274079851790721257797144758322315908160396257811764037237817632071521432200871554290742929910593433240445888801654119365080363356052330830046095157579514014558463078285911814024728965016135886601981690748037476461291163877376.

The above number has 241 digits and its value is exact, not approximate. The same is true of the number below, but it is not quite the same number.

2^{800}- 1 = 6668014432879854274079851790721257797144758322315908160396257811764037237817632071521432200871554290742929910593433240445888801654119365080363356052330830046095157579514014558463078285911814024728965016135886601981690748037476461291163877375.

6) Let L represent the lower number and let U represent the upper number. U - L = ___

7) The last two digits of U - 1 are 75. What are the last two digits of U? ____

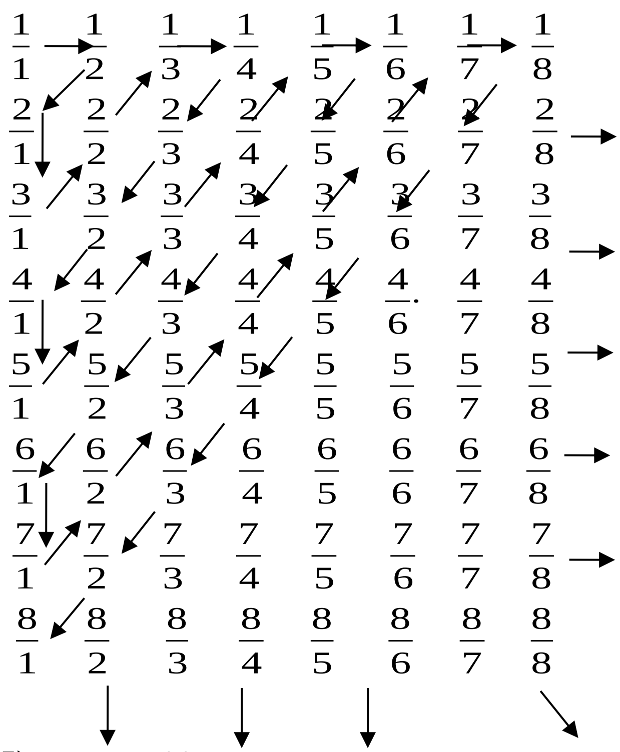
8) Which is smaller, the reciprocal of L or the reciprocal of U? ____________________

A BIGGER INFINITE? **Unit 32

Answer 1 – 5

1) 11, 20 **2)** 20, 11 **3)** 100, 17

4)



The outer arrows indicate that the array and the inner arrows continue without ending.

5) Any pair of fractions that reduce to the same number.

Resume at top of Page 4 of this unit.

9) Which is smaller, $2^{(800 - 1)}$ or $(2^{800} - 1)$? __________

10) Tell whether each is odd or even:

a) 344×10^{420} _____ b) 345×10^{420} _____ c) 344×10^{121} _____ d) 345×10^{121} _____

11) Odd or even? a) 2^{400} _____ b) 2^{401} _____ c) 2^{500+1} _____ d) $2^{500} + 1$ _____

*12) 10^{100} is larger than the number of sub-atomic particles in the universe! So, 2^{800} is (much larger/ much smaller) ________________________than the number of sub-atomic particles in the universe. (Unit 1 tells how many digits in 10^{100})

Correct Exercises 6 – 12 now.

A BIGGER INFINITE? **Unit 32

Text Resumed

$$\frac{666801443287985427407985179072125779714475832231590816039625781176403723781763207152143220087155429074292991059343324044588880165411936508036335605233083004609515757951401455846307828591181402472896501613588660198169074803747646129116387737 6}{666801443287985427407985179072125779714475832231590816039625781176403723781763207152143220087155429074292991059343324044588880165411936508036335605233083004609515757951401455846307828591181402472896501613588660198169074803747646129116387737 7}$$

This is a fraction

(not 6)

13) Will the big fraction above occur in the *extended* diagram on page 5? Y/N ____

14) The fraction for 13) could be expressed as $\frac{2^{800}}{2^{800}+1}$. Which is true, a or b?____

a) This fraction would appear in the 800th row.

b) This fraction would appear in the 2^{800}th row.

It is not easy to get a "feeling" or "sense" of how vastly large these numbers are. The correct answer to ex. 19) is b), the 2^{800}th row!! Any fraction in the 800th row would have only 800 for a numerator. But 2^{800} is that number we said is much larger than the number of sub-atomic particles in the universe and it has 241 digits. So the numerator is:

666801443287985427407985179072125779714475832231590816039625781176403723
781763207152143220087155429074292991059343324044588880165411936508036335
605233083004609515757951401455846307828591181402472896501613588660198169
0748037476461291163877376 = 2^{800}.

"Ten to the 60 millionth power"

Yet, we can write much larger finite numbers. Let's suppose we are in the $10^{60,000,000}$th <u>column</u> of Cantor's array. (Yes $\mathbf{10}^{60,000,000}$, not $\mathbf{2}^{60,000,000}$). Please note that this is not merely the 60,000,000th column. It is the $10^{60,000,000}$th column. Remember from Unit I that the number $10^{60,000,000}$ is 1 with 60 million zeros after it. To print out 1, followed by 60 million zeros, we need a surprising amount of space.

A BIGGER INFINITE? **Unit 32

If we fill a 1 inch thick book with 250 pages of 4000 zeros each (say 50 lines of 80 zeros each), we will have a book with about 1,000,000 zeros.

15) A stack of how many such books would give us 60,000,000 zeros? ____________

16) The denominator of every fraction identifies row or column?____________

17) Suppose that we are in the $10^{400,000}$th row and the $10^{400,000}$th column. We might decide not to count this fraction. Why might we? ____
____

18) The first fraction in exercise 4, having the same denominator as numerator, joins many other fractions having the same problem and they all fall in a line (not drawn) from the upper left of the array toward the lower right. Such a line is called a ______________.

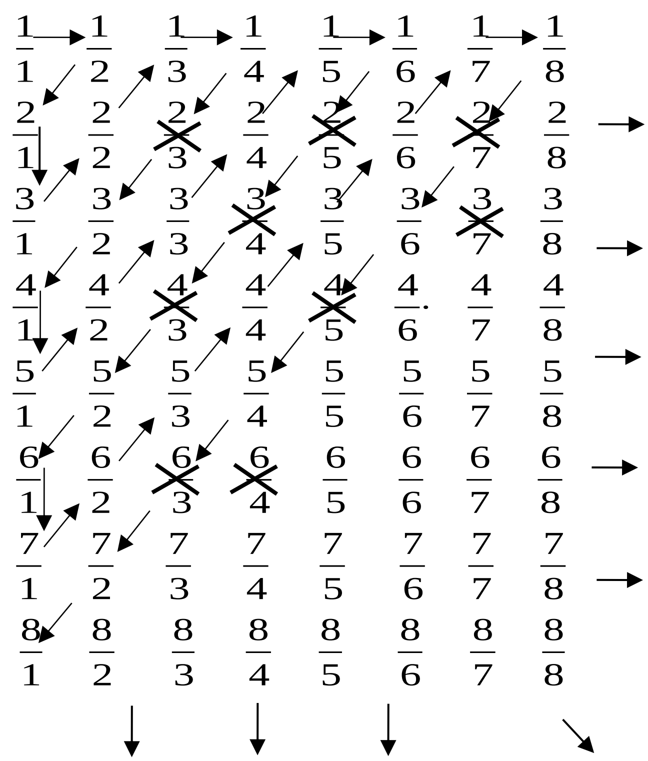
19) The reduced value of every fraction on the main diagonal is ______.

20a) How many values are in the first (top) row of Cantor's array? ________
b) How many values are in the first (left) column of the array? _________
c) How many values are in the diagonal from the upper left toward the lower right?_______

Your correct answers of $\aleph_0$ for both 20a) and 20b) guarantee that there are at least $\aleph_0$ fractions in the set of rational numbers, the infinitely extending rectangle that we have been working on.

Note: Exercise 20c) might have been worded "… diagonal to the lower right?" But this would suggest a destination, a "place" where the diagonal might stop. But a diagonal is fully as unending as a row or column, and therefore has no destination, only a "direction".

A BIGGER INFINITE? **Unit 32



> **Note:** It can be easily seen in the drawing to the left that, even if we did not cross out the unreduced fractions, the trail would not be altered and the 1 – 1 correspondence would still hold. These fractions would thereby add to the mix but the cardinal number Aleph null ($\aleph_0$) would be correct for this "enlarged" set too.
>
> $\aleph_0 + \aleph_0 = \aleph_0$

There cannot be any doubt that this scheme on page 6 **counts them all** by showing there is a reduced fraction for every counting number and there is a counting number for every reduced fraction, because neither set ends and a matching plan is established. Thus the rational numbers are in one-to-one correspondence with the counting numbers. As shown by the genius of Cantor's original thinking, **the cardinal number of the set of rational numbers is $\aleph_0$**.

We saw earlier that fractions are said to be *densely packed*. Between any two rationals on the number line there is always a third. Just add the two and divide by 2, thereby finding their average which surely lies between them. In fractions **there is no next number**. Who can be blamed for saying there are more fractions than counting numbers? It certainly seems so. To say "Either there are more fractions than whole numbers or else there is something wrong with Cantor's proof," might be an understandable reaction. Georg Cantor was becoming famous but not necessarily happy. Unfriendly disagreement among some mathematicians quickly greeted his new work.

It seems incredible that the rational fractions do not provide a "Yes" to the title of this unit.

A BIGGER INFINITE? **Unit 32

Answers 6 - 20

6) 1 **7)** 76 **8)** The reciprocal of U **9)** 2^{800-1}

Question: 2^{800-1}, $2^{800}-1$ and 2^{400}**.** Above, one of these is exactly half of 2^{800}. Which one?_________

10) All are even

11) a) to c) are even; d) is odd.

12) Much larger. 2^{800} has 241 digits (pg.4); 10^{100} has 101digits (Unit 1)

Resume at the top of Page 6 of this unit to continue.

13) yes **14)** b, 2^{800}th row **15)** 60 books **16)** column

17) The fraction is in the main diagonal and so reduces to 1, already counted.

18) Diagonal **19)** 1 **20a)** $\aleph_0$ **b)** $\aleph_0$ **c)** 1

The answer to answer 9's question is 2^{800-1}.

<u>Handling a fraction</u> like the one below can give us some sense of the very large: →

$$\frac{6668014432879854274079851790721257797144758322315908160396257811764037237817632071521432200871554290742929910593433240445888801654119365080363356052330830046095157579514014558463078285911814024728965016135886601981690748037476461291163877376}{6668014432879854274079851790721257797144758322315908160396257811764037237817632071521432200871554290742929910593433240445888801654119365080363356052330830046095157579514014558463078285911814024728965016135886601981690748037476461291163877377}$$

(not 6)

This is a fraction

21) The size of the above fraction is very close to 1. So is its reciprocal. Which is closer? ______

A BIGGER INFINITE? **Unit 32

Reflections on This Unit

- ✓ No matter how many numbers are beyond 1, just as many are between 0 and 1.
- ✓ 0 has no reciprocal.
- ✓ There is no next number in the rationals--no two rationals are consecutive.
- ✓ There is no largest rational number; there is no smallest, either.
- ✓ The reciprocal of any number larger than 1, is between 0 and 1.
- ✓ The reciprocal of any number between 0 and 1 is greater than 1.
- ✓ The number 2^{800} has 241 digits, yet its reciprocal lies between 0 and 1.
- ✓ There are $\aleph_0$ natural numbers, negative numbers, rational numbers, evens, numbers divisible (evenly) by 29,429,000, primes and infinitely(?) many other infinite proper subsets of the rational numbers.
- ✓ If we fail to cross out the unreduced fractions when matching 1-1 against the natural numbers, the proof still holds.
- ✓ In fact, we would be proving that $\aleph_0$ is not only the cardinal number of the rationals but also of the rationals together with their unreduced duplicates in the list.

The answer to exercise 21 on previous page is “The number is closer to 1, not it’s reciprocal”.

ARE WE THERE YET? **Unit 33

No, Elizabeth. We are not there yet. We have talked only about some large finite numbers like $10^{60,000,000}$, but this is not even a drop in the ocean when $\aleph_0$ is in the conversation. We are in the middle of trying to determine which infinite sets of numbers on the number line are countable, that is, numbers which match the counting numbers 1-to-1, and which, if any, exceed $\aleph_0$. We did skip over some sets without bothering to show or **prove** their size. One such set was the set of integers:

I = { . . . -3, -2, -1, 0, 1, 2, 3 . . . }

Are they countably infinite? How do you count them? There seems to be no place to begin because the integers have no first number. If you start matching with the counting numbers at the integer 0 you can't expect to *end,* so you could never go back to count the opposite way into the negatives. Cantor was not bothered by this. He said simply to start at 0, but *alternate* the positive and negative integers as you go along.

1) Complete the beginning nine matches of Cantor's plan with arrows connecting the natural numbers and their matching integers.

1	2	____	____	____	____	____	____	____
↕	↕	↕	↕	↕	↕	↕	↕	↕
0	+1	-1	+2	-2	____	____	____	____

*2) According to the patterns in the plan above (Cantor called this "Interlacing"), show which integers match the counting numbers at right. There is only one correct answer set.

693	694	695	696	697	698
↕	↕	↕	↕	↕	↕
____	____	____	____	____	____

The bottom row of each of the problems 1 & 2 shows the integers after you interlaced them into a different order so they could be matched with the counting numbers. **The answers to 1 and *2** are on the next page. Correct them now and **return**.

Resume here after correcting exercise 1 and 2.

We pause here for a moment to give some attention to the kind of work that Cantor did. He chose a mathematical topic, the infinite, which lay undeveloped by scholars of his time. Some shunned the topic and declared that the infinite did not exist in any formal way and mathematics should leave infinity as a dead issue. Cantor, on the other hand, embarked upon constructing a formal study to place “transfinites” on firm mathematical ground, and organize its structure into a valid system worthy of study and further discovery by professional mathematicians. We do not approach this with such credentials, but, with persistence and diligence, we can see quite deeply into this work of genius which offers us life-enriching experiences. There are many astounding ideas and a vast ground for surprises in Cantor’s Theory of Transfinites.

Let **n** be any natural number

$$\aleph_0 + \mathbf{n} = \aleph_0 \qquad \mathbf{n} \times \aleph_0 = \aleph_0$$

$$\aleph_0 \times \aleph_0 = \aleph_0 \qquad \aleph_0^{\mathbf{n}} = \aleph_0$$

Answers 1 – 2

1)

1	2	3	4	5	6	7	8	9
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
0	+1	-1	+2	-2	+3	-3	+4	-4

2)

693	694	695	696	697	698
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
-346	+347	-347	+348	-348	+349

Return to page 1 after correcting.

Exercise 2) shows that the interlacing pattern will work unceasingly if needed.

Let $\aleph_0 + \aleph_0 = \aleph_0$ represent the statement "When two countably infinite sets are added, the sum is also countably infinite".

ARE WE THERE YET? **Unit 33

Numbers Considered So Far.

Integers { Zero / Negative whole numbers / Positive whole numbers, or naturals }

Rational Numbers { Integers / Ratios of one integer to another }

Nowhere above is there a mention of decimals. How do the decimals fit in the above arrangement? Some of them belong in the category "Ratios of one integer to another", such as 3/4 = .75, but not all decimals fit in that category.

After the rational numbers comes a classification called Algebraic Numbers. Not fair! Your work is supposed to be non-algebraic. True, but this is fairly easy and brief**.** Algebraic numbers are those which are solutions to algebraic equations**:** 2y + 3 = 13. This says 2 times some number, then add 3 to get 13.

3) What number works when it replaces the y, multiplied by 2, etc.? ______ You can solve some algebraic equations by staring at them until a solution occurs to you.

Here are some of those. Give solutions in any convenient form.

4) 2y + 4 = 13, y = _____ 5) y + .34 = .46, y = ____ *6) 2y = $.\overline{08}$, y = ____

You probably got them all right if you stared at them long enough.

7) Do all of your answers fit one or more of the classifications of Algebraic Numbers? _______

Answers 3 – 7

3) 5 **4)** 4 1/2 or 4.5 or 9/2 **5)** .12 or 3/25 **6)** $.\overline{04}$ or 4/99 **7)** Yes

ARE WE THERE YET? **Unit 33

Any surprises? In exercises 3 - 6, all of the forms given in the answers are okay, but *the last* form in each case -- 5, 9/2, 3/25, 4/99 -- qualifies all of them as rational numbers. The whole number 5 is a rational number**:** 5 = 5/1. This is shown in the classification at the top of the previous page where integers are included as rational numbers or exercise 6, $2\overline{y}$ = .08. Maybe you got **.**04, but maybe not 4/99. Do 4 ÷ 99 now. Calculator O.K. There is probably a calculator on a computer you might use.

Use your calculator to verify the repeating decimal for the following fraction:
5/37 = $.1351\overline{351}$ **. . .** (8 place calculator); 5/37 = $.135135\overline{135}$ **. . .** (10 place calculator)
Simplest answer: **.**$\overline{135}$. Some denominators to experiment with, especially on a computer's calculator because of their usual 20^+ digit readout, are primes with 2 or 3 digits.

Calculator OK**:** Find the repeating decimal equivalents (Simplest form answers)

8) $\frac{28}{99} =$ 9) $\frac{28}{999} =$ 10) $\frac{31}{33} = \frac{93}{\square} =$ 11) $\frac{47}{9999} =$

You can probably see a pattern already. Now let's see it backwards. Give the fraction.

12) $.\overline{29} =$ 13) $.\overline{39} =$ (Reduce) 14) $.\overline{123} =$ (Reduce) 15) $.\overline{324} =$ (Reduce)

Answers are on next page.

What we really need to know is---**are all repeating decimals rational numbers**? You have been using patterns, but what magic in denominators 99 or 999 (etc.) mysteriously produces a repeating decimal? Will it always work? Understanding the magic can help in understanding whether all repeating decimals are rational numbers.

We will use **exercise 12** to investigate the magic. Don't try yet to understand *why* each step is taken. Just see what each step does, and that we finally accomplish what we want, namely that $.\overline{29}$ = 29/99.

First: Multiply **.**2929292929 **. . .** by 100. This moves the point (or the number) 2 places and gives 29**.**29292929 **. . .** (Of course the new number is 100 times the old.).

Next: Subtract our original decimal from 29**.**29292929 **. . . .**

ARE WE THERE YET? **Unit 33

16) We now have:

Subtract 1 of the decimals from100 of the decimals and you get ___ of them.

100 of the decimals = 29.29292929 . . .
− 1 of the decimals = − .29292929 . . .
☐ of the decimals = ☐

same

As on the left side, subtract the bottom from the top and you can see that the decimal part gets wiped out by the subtraction.

The shortcut!

*17) Now if 99 of the decimals = 29, then 1 decimal = 29 ÷ ____ = 29/____

18) If the repetend (the repeating piece 29) in exercise 16 had been 3 digits instead of 2, then multiplying by 100 would not have set things up for a wipeout subtraction. We would have needed to multiply by __________, not 100.

19) If the repetend (the repeating piece) in exercise 16 had five digits, we would have multiplied not by 100 but instead by __________ = 10^{----}

*20) If the repetend in a repeating decimal contained $\boxed{\mathbf{n}}$ digits (starting at the point) the first step in the above procedure would be to multiply the decimal by _____. (Use an exponent)

You are not expected to *use* the procedure above. Just study it through so you can agree that everything in the procedure is legal. Also, and of real importance, see that the procedure can accommodate *any* number of digits in the repetend and it does not matter what they are. This means that the procedure and its shortcut you have just used will change *any* repeating decimal to a fraction. **So,** all repeating decimals are rational numbers and the set of rational numbers has the cardinal number $\aleph_0$.

Answers 8 – 15

8) $.\overline{28}$ **9)** $.\overline{028}$ **10)** $\boxed{99}$, $.\overline{93}$ **11)** $.\overline{0047}$ **12)** $\frac{29}{99}$

13) $\frac{13}{33}$ **14)** $\frac{41}{333}$ **15)** $\frac{12}{37}$

Return to Page 4 to continue

Answers 16 – 20

16) 99 - - 29 **17)** 99 - - 99 **18)** 1000 **19)** 100,000, ^{5} **20)** 10^{n}

ARE WE THERE YET? **Unit 33

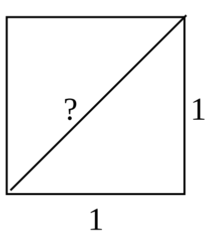
Integers: { Zero; Negative whole numbers; Positive whole numbers, or naturals }

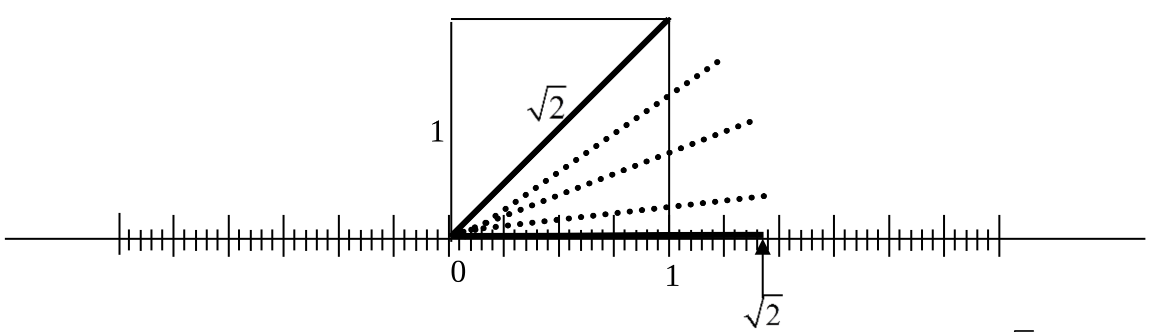
Rational Numbers: { Integers; Ratios of one integer to another }

Algebraic Numbers: { Rational numbers; Solutions to certain algebraic equations }

You have done well if you have come this far in Unit 33 with reasonably good understanding. In the classification above we have added *algebraic numbers* to the version on page 3. We have already dealt with a few of these, but it turns out that *all* of the algebraic numbers we saw were also rational numbers – many were repeating decimals.

21) Here is an easy repeat example. Use Pythagoras to find the length of the diagonal of a square whose side is 1 unit long. Show work. The answer comes out as the square root of a definite number: $\sqrt{?}$. Leave it in that form for now. ______





This figure shows the answer to exercise 21, and more. The diagonal length is $\sqrt{2}$ units.

ARE WE THERE YET? **Unit 33

We then let the diagonal, hinged at 0, drop down to the number line. This establishes where $\sqrt{2}$ is on the number line. And it shows beyond doubt that there *is* a point on the number line called $\sqrt{2}$.

There hardly seems to be room for many more numbers on the number line. We have already said that the rational numbers are densely packed -- no rational number has a next rational. Given two rationals, no matter how close, there is always one (at least) rational between them -- their average. But here we are, inserting yet another kind of number.

The ancient Greeks were quite troubled by a number like $\sqrt{2}$. They could not find an exact value for it. They found values like $\sqrt{\frac{49}{25}}$. Notice how close $\frac{49}{25}$ is to 2. Eventually, it was thought, they could find two numbers with perfect square top and bottom but also with an *exact* ratio 2 to 1. Because the fraction $\frac{49}{25}$ has a perfect square in both top and bottom, its square root is rational. But we have the same problem the Greeks had. We cannot find an exact value either. $\sqrt{2} \approx$ 1.4142135623730950488016887242097 according to the computer's calculator. ($\approx$ means "approximately equals"). Now, if you squared that number on your calculator, you might get 2.00000000000 . . . *but* we know that the product of the above number and <u>itself</u> is going to end in 9, not 0 (because of 7 x 7). The calculator would fool us by rounding.

What we have, then, is a non-ending *and* non-repeating decimal for $\sqrt{2}$. It is not like $\sqrt{\frac{49}{25}} = \frac{7}{5}$; it cannot be expressed as a ratio of two naturals, so $\sqrt{2}$ is not rational. This number is *irrational*, meaning not rational. Hooray! At last we might have escaped $\aleph_0$ along with millions of other such irrationals (non-ending, non-repeating decimals).

Sorry. Yes, there are millions of other irrationals which are solutions to other equations. But Cantor, by some wonderfully clever grouping of all solutions to algebraic equations, was able to show that algebraic numbers are countable. So all algebraic numbers, which also include the rationals, which include the integers and all that they include, still have the cardinal number $\aleph_0$.

$$\aleph_0 + \aleph_0 + \aleph_0 + \aleph_0 = \aleph_0 .$$

MENTAL MATH 3, FRACTIONS — Unit 34

Study patterns: $\frac{1}{10}+\frac{1}{7}=\frac{17}{70}$ $\quad\frac{1}{3}+\frac{1}{20}=\frac{23}{60}$ $\quad\frac{1}{11}+\frac{1}{9}=\frac{20}{99}$ $\quad\frac{1}{5}+\frac{1}{7}=\frac{12}{35}$

Do these mentally:

1 $a)\frac{1}{5}+\frac{1}{3}=$ $\quad b)\frac{1}{2}+\frac{1}{7}=$ $\quad c)\frac{1}{10}+\frac{1}{9}=$ $\quad d)\frac{1}{9}+\frac{1}{4}=$

2 $a)\frac{1}{5}+\frac{1}{11}=$ $\quad b)\frac{1}{7}+\frac{1}{20}=$ $\quad c)\frac{1}{100}+\frac{1}{3}=$ $\quad d)\frac{1}{11}+\frac{1}{8}=$

Notice subtraction

Correct to here and then study the following:

$\frac{1}{100}+\frac{2}{7}=\frac{207}{700}$ $\quad\frac{1}{10}+\frac{2}{3}=\frac{23}{30}$ $\quad\frac{1}{12}+\frac{4}{5}=\frac{53}{60}$ $\quad\frac{5}{11}-\frac{1}{300}=\frac{1489}{3300}$

Do these:

3 $a)\frac{1}{100}+\frac{2}{3}=$ $\quad b)\frac{9}{11}+\frac{1}{200}=$ $\quad c)\frac{1}{8}+\frac{3}{5}=$ $\quad d)\frac{3}{5}+\frac{1}{2}=$

4a) $\frac{4}{5}+\frac{1}{\bigcirc}=\frac{\square}{10}$ $\quad b)\frac{1}{7}+\frac{3}{\bigcirc}=\frac{\square}{28}$ $\quad *c)\frac{9}{10}-\frac{\bigcirc}{7}=\frac{53}{\square}$ $\quad d)\frac{10}{11}-\frac{1}{7}=$

Correct above and study the patterns below.

$\frac{6}{7}+\frac{7}{100}=\frac{649}{700}$ $\quad\frac{3}{100}+\frac{2}{3}=\frac{209}{900}$ $\quad\frac{7}{10}+\frac{3}{40}=\frac{310}{400}=\frac{31}{40}$

5 $a)\frac{2}{7}+\frac{3}{10}=$ $\quad b)\frac{6}{7}+\frac{9}{10}=$ $\quad c)\frac{3}{7}+\frac{2}{5}=$ $\quad d)\frac{11}{30}+\frac{2}{7}=$

6 $a)\frac{3}{20}+\frac{3}{2}=$ $\quad b)\ \frac{\square}{8}+\frac{6}{\bigcirc}=\frac{63}{40}$ $\quad c)\ \frac{5}{\square}+\frac{\bigcirc}{3}=\frac{29}{21}$

<u>Subtracting Fractions:</u>

Samples: $\frac{4}{7}-\frac{1}{10}=\frac{33}{70}$ $\quad\frac{10}{11}-\frac{1}{10}=\frac{99}{110}$ $\quad\frac{4}{7}-\frac{2}{5}=\frac{6}{35}$

$13^2 = 169$
See exercise 8b

7 $a)\frac{8}{11}-\frac{1}{100}=$ $\quad b)\frac{4}{7}-\frac{1}{5}=$ $\quad c)\frac{4}{7}-\frac{2}{5}=$ $\quad d)\frac{2}{11}-\frac{1}{100}=$

8 $a)\frac{20}{21}-\frac{19}{20}=$ $\quad b)\frac{13}{14}-\frac{12}{13}=$ $\quad c)\frac{11}{12}-\frac{10}{11}=$ $\quad d)\frac{30}{31}-\frac{29}{30}=$

MENTAL MATH 3, FRACTIONS Unit 34

Answers

1 $a)\frac{1}{5}+\frac{1}{3}=\frac{8}{15}$ $\quad b)\frac{1}{2}+\frac{1}{7}=\frac{9}{14}$ $\quad c)\frac{1}{10}+\frac{1}{9}=\frac{19}{90}$ $\quad d)\frac{1}{9}+\frac{1}{4}=\frac{13}{36}$

2 $a)\frac{1}{5}+\frac{1}{11}=\frac{16}{55}$ $\quad b)\frac{1}{7}+\frac{1}{20}=\frac{27}{140}$ $\quad c)\frac{1}{100}+\frac{1}{3}=\frac{103}{300}$ $\quad d)\frac{1}{11}+\frac{1}{8}=\frac{19}{88}$

3 $a)\frac{1}{100}+\frac{2}{3}=\frac{203}{300}$ $\quad b)\frac{9}{11}+\frac{1}{200}=\frac{1811}{2200}$ $\quad c)\frac{1}{8}+\frac{3}{5}=\frac{29}{40}$ $\quad d)\frac{3}{5}+\frac{1}{2}=\frac{11}{10}$

4a) $\frac{4}{5}+\frac{1}{2}=\frac{\boxed{13}}{10}$ $\quad b)\frac{1}{7}+\frac{3}{4}=\frac{\boxed{25}}{28}$ $\quad *c)\frac{9}{10}-\frac{1}{7}=\frac{53}{\boxed{70}}$ $\quad d)\frac{10}{11}-\frac{1}{7}=\frac{59}{77}$

5 $a)\frac{2}{7}+\frac{3}{10}=\frac{41}{70}$ $\quad b)\frac{6}{7}+\frac{9}{10}=\frac{123}{70}$ $\quad c)\frac{3}{7}+\frac{2}{5}=\frac{29}{35}$ $\quad d)\frac{11}{30}+\frac{2}{7}=\frac{137}{210}$

6 $a)\frac{3}{20}+\frac{3}{2}=\frac{66}{40}\, or\, \frac{33}{20}$ $\quad b)\ \frac{\boxed{3}}{8}+\frac{6}{5}=\frac{63}{40}$ $\quad c)\frac{5}{\boxed{7}}+\frac{2}{3}=\frac{29}{21}$

7 $a)\frac{8}{11}-\frac{1}{100}=\frac{789}{1100}$ $\quad b)\frac{4}{7}-\frac{1}{5}=\frac{13}{35}$ $\quad c)\frac{4}{7}-\frac{2}{5}=\frac{6}{35}$ $\quad d)\frac{2}{11}-\frac{1}{100}=\frac{189}{1100}$

8 $a)\frac{20}{21}-\frac{19}{20}=\frac{1}{420}$ $\quad b)\frac{13}{14}-\frac{12}{13}=\frac{1}{182}$ $\quad c)\frac{11}{12}-\frac{10}{11}=\frac{1}{132}$ $\quad d)\frac{30}{31}-\frac{29}{30}=\frac{1}{930}$

INTERMISSION **Unit 35

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Do all solutions on your calculator and don't correct any until you have done 1 - 12.
Keep watch for unexpected changes in answer patterns that might signal a mistake.

Keep as many digits in your answers as your calculator provides.

$(1 + \frac{1}{n})^n$ when n = 1, is $(1 + \frac{1}{1})^1 = (1 + 1)^1 = 2^1 = 2$

$(1 + \frac{1}{n})^n$ when n = 2, is $(1 + \frac{1}{2})^2 = (\frac{3}{2})^2 = \frac{9}{4} = 2.25$

(Reciprocal key) Your calculator key might say ^ or x^y

1) Using **3** as n**,** enter into your calculator as follows 1, +, 3, $\frac{1}{x}$, =, y^x, 3, =

Answer is 2.37037037 ____________________

2) $(1 + \frac{1}{n})^n$ when n = 10, is ____________________

Use exercise 1 as a guide.

3) $(1 + \frac{1}{n})^n$ when n = 20, is ____________________

4) $(1 + \frac{1}{n})^n$ when n = 100, is ____________________

Continue in the same way and notice how the answers begin to stabilize.

5) n = 200 ____________________ 9) n = 20,000 ____________________

6) n = 500 ____________________ 10) n = 50,000 ____________________

7) n = 1000 ____________________ 11) n = 100,000 ____________________

8) n = 10,000 ____________________ 12) n = 1,000,000 ____________________

Correct answers 2 – 12 now. Answers on the next page.

We increased the values of n *more* rapidly with each example. But, the values of the expression $(1 + \frac{1}{n})^n$ increased more *slowly* each time.

For exercises 10 to 11, n increased from 50,000 to 100,000, while the answers increased only .000013613. Compare this to the changes for n = 1 to n = 2 in the illustrative problems at top.

INTERMISSION **Unit 35

There is a *limit* above which the value of $(1 + \frac{1}{n})^n$ will not go. You have met the idea of limits before in Achilles and the Tortoise. Even as n grows *infinitely* large, $(1+\frac{1}{n})^n$ stabilizes close to 2.7.

The expression for **e** is not the solution of any algebraic solution. $\sqrt{2}$ is a solution to the algebraic equation $x^2 = 2$, but **e** is not a root of any such equation, so **e** is said to be a **transcendental** number. More later.

$\lim_{n\to\infty}(1 + \frac{1}{n})^n =$ is shorthand for exactly what you have been doing in exercise 1 - 12.

You can read it roughly "As n gets larger and larger, $(1 + \frac{1}{n})^n$ gets closer and closer to **e**:

e = 2.71828182845904523536028747135266 . . . .

e ≈ 2.71828182845904523536028747135266 (≈ means approximately =)

Note the interesting pattern after the 2.7 for the next 14 digits.

Answers 1 – 12

1) $(1 + \frac{1}{n})^n$ when n = 3, is 2.37037037

2) $(1 + \frac{1}{n})^n$ when n = 10, is 2.59374246

3) $(1 + \frac{1}{n})^n$ when n = 20, is 2.653297705

4) $(1 + \frac{1}{n})^n$ when n = 100, is 2.70481383

If you have considerable difficulty getting answers correct and you are sure that you are following the entries given on page 1, try using a computer's calculator. In Windows the calculator is under Accessories. As a calculator exercise, this is a demanding one and you might want to seek help. A computer's calculator will give different digits than the calculator's ending digits. This effect increases starting with n=1000.

Every answer from 4 onward starts with 2.7

Continue in the same way but notice how the numbers begin to stabilize.

5) n = 200 2.711517123

6) n = 500 2.715568521

7) n = 1000 2.716923933

8) n = 10,000 2.718145936

9) n = 20,000 2.718213889

10) n = 50,000 2.718254646

11) n = 100,000 2.718268303

12) n = 1,000,000 2.718281378

INTERMISSION **Unit 35

A Matter of Interest

13) $1000 can be increased by, say 6%, by multiplying and adding: $1000 x .06 = $60

$1000 + $60 = $_______

14) A shorter way to do this is to multiply $1000 x 1.06.

In your answer, note that the 1 in the 1.06 gives back the 1000 and the 6 in the 1.06 gives back the 60 (after the point is moved).

$1000
x 1.06
6000
1000
$ _______

Answer: Don't forget the decimal point.

15) Use the 1.06 method to increase each of these by 6% (Calculator O.K.):

a) $200 _______ b) $9000 _________ c) $120,000,000 ____________________

16) If you wanted to increase $9000 by 6% three times, you could do ex. 15b), then increase that answer by 6%, and finally increase *that* answer by 6%.

a) **T/F** This could be represented by: (($9000 x 1.06) x 1.06) x 1.06 ___

b) or **T/F** it could be done this way: $9000 x 1.06^3 ___

c) Do it. _________

*17) It could be argued that the first increase, $540, should be given three times. But if a bank did that to a customer, the bank would be paying the customer **simple interest.** The way you did it is called **compound interest** because the previous interest is included in the **new amount** each time. You get a little more money this way. How much more over a period of 3 years, if interest is compounded once per year?__________

18) Often a bank will offer interest compounded not each year (annually), but every six months (semi-annually). If the rate is still 6% per year they will not pay you that much twice a year. They will pay you 3% every six months. It sounds reasonable but will it make any difference? It will make a difference if interest is compounded. Give the letter of the choice that you think would be the best deal for the customer for a three year loan.

a) $9000 x 1.06^3 b) $9000 x 1.03^6 c) $9000 x 1.03^3 d) $9000 x 1.06^6 ____

19) Give the letter of the best choice that you think the bank would offer: ____ It is pretty clear that the bank will not pay a full year's worth of interest twice a year. But it is also clear that the bank must do *something* for you every six months.

20) In the choices above, the exponent tells the number of times something happens and the decimal (%) tells the rate of interest to be paid each time. **T/F** ____

INTERMISSION **Unit 35

Correct exercises 13 – 20 now.

21) The solutions to this problem are at the bottom of the page. Peek at them as little as possible as you work. Find the amount on $2000 at an interest rate of 12% per year compounded:

a) annually ____________ b) semi-annually _____________ c) quarterly _____________

**d) monthly ____________ **e) daily (*Use 360 days per year to simply*) _____________

Complete the checking of exercise 21 now.

Probably you noticed that the calculator operations resembled those on page 1, the expression for **e**. Now we will use some numbers for interest rate and amount which would shock a bank manager but are useful mathematically.

They are 100% = interest rate and $1 = amount. The important effect here is that 100% = 1 and amount also = 1, making the computation simpler and revealing a surprising fact.

*22) Use the new rate and amount to compute the new $1 amount when compounded:

a) Monthly _____________ b) Daily _______________ c) Hourly____________________

Answers 13- 22

13) $1060 **14)** $1060.00 or $1060

15a) $212 **b)** $9540 **c)** $127,200,000

16a) T **b)** T **c)** $10719.14

17) $99.14 **18)** d **19)** b **20)** T

21) You can enter the numbers in your calculator in the order that they appear in the original expression except for part e) where the numbers are too large. There, multiply by 2000 last.

a) $2000 x 1.12 = $2240 **b)** $2000 x 1.06^2 = $2247.20

c) $\$2000\times\left(1+\frac{.12}{4}\right)^4$ = \$2000 x $(1.03)^4$ = \$2000 x 1.12550881 = \$2251.02

d) $\$2000\times\left(1+\frac{.12}{12}\right)^{12}$ = \$2000 x 1.01^{12} = \$2000 x 1.12682503 = \$2253.65

e) $\$2000\times(1+\frac{.12}{360})^{360}$ = \$2000 x 1.000333333^{360} = \$2000 x (1.127474306) = \$2254.95

Looking at the increases in the answers shows that they are getting larger and larger but more slowly each time. This suggests that you can leave your money in the bank and it will always grow more each day, but at a *slowing increase.*

22a) \$1 x $(1+\frac{1}{12})^{12}$ = 1 x $(1 + .083333333)^{12}$ = 2.61303529 = \$2.61

b) \$1 x $(1 + \frac{1}{360})^{360}$ = 1 x $(1 + .002777778)^{360}$ = 2.714516027 = \$2.71

c) \$1 x $(1 + \frac{1}{360\times 24})^{360 \text{ x } 24}$ = 1 x $(1 + .000115741)^{8640}$ = 2.71812453 = \$2.72

Notice that exercise 22a, 1 x $(1 + \frac{1}{12})^{12}$ could be written $(1 + \frac{1}{12})^{12}$. This drops the 1 x, because 1 x any expression of numbers <u>is the same as</u> that expression. This makes the computation look just like that on the first two pages where we were getting closer and closer to **e.** From page 2, enter the value of **e** in your calculator. Raise that value to the .12 power. Compare with example 21e) above; look at the next-to-last number in that 12% interest problem. Surprised?

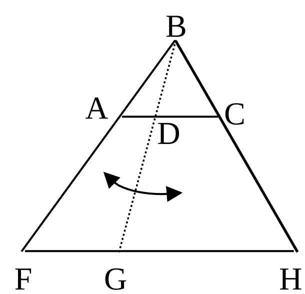
Starting on page 99 in his book *Mathematical Mysteries*, Calvin Clawson gives us some excellent insights into **e** and a few of its many interesting applications in mathematics. They include compound interest, the amount of grain a farmer needs to plant in order to have his crop *and* enough seed to plant the following year, and establishing the time of death by computing rate of heat loss from the body. See #6 on our book list.

INTO THE ABYSS (THE UNCOUNTABLE) ***Unit 36

Abyss: An immeasurably deep gulf, or great space. (Webster)

Anyone having or getting a good middle school background in mathematics can have considerable depth of understanding in Georg Cantor's life work. When Cantor called a set "uncountable" he was not talking about a shortage of time or patience. He meant that there are *too many objects in the set to be counted*. To find the size of a set of numbers that is uncountable requires some method different from listing numbers in a 1 to 1 match with the natural numbers. More later.

In the triangle at right (follow it as you read), the lighter line segment BG is "hinged" at B**.** BG swings back and forth through positions from BF to BH. There will always be a point like D corresponding to G and always a point like G corresponding to D, that mark the two intersections of the swinging segment BG with horizontal segments AC and FH.



All corresponding pairs of points like those at D and G will continuously match 1 – 1 as segment BG swings. This suggests that a smaller line segment (AC) has as many points as a longer segment (FH). We have seen such unexpected conclusions before and will see more.

In Unit 30 you worked with the number **e**, a transcendental number. You worked out the expression $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$ for larger and larger values of n to get closer and closer approximations to **e**. That repeated "working out" seems to make **e** more like a process than a number. It *transcends***,** or in some way *surpasses* ordinary numbers**.**

π is another transcendental number with which you are familiar but probably knew little about as a transcendental. It is often seen in dealing with a circle. One mathematician in 1853, William Shanks, computed π to more than 900 decimal places but it was found later that he had made an error soon after 500 decimal places. (See p. 107 of #14 on book list.)

Two expressions for π appear on page 138 of The World of Mathematics; see #12 on our book list. The expressions are:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \quad and \quad \frac{\pi}{4} = \frac{2\ x\ 4\ x\ 4\ x\ 6\ x\ 6\ x\ 8\ x\ \ldots .}{3\ x\ 3\ x\ 5\ x\ 5\ x\ 7\ x\ 7\ x\ \ldots .}.$$

Again, the transcendental number π **se**ems more like a *process* than a number.

INTO THE ABYSS (THE UNCOUNTABLE) ***Unit 36

Integers
-5, 0, 332
- Zero
- Negative whole numbers
- Positive whole numbers, naturals

↓

Rational Numbers
2/3, $.\overline{3}$, -27/5
- Integers
- Ratios of one integer to another

↓

Algebraic Numbers
$\sqrt{2}$, 3/11, $-\sqrt{10}$, 493…
- Rational numbers
- Solutions to algebraic equations,
- non-ending *and* non-repeating(irrationals)

↓

Real Numbers
- Algebraic numbers
- Transcendental numbers**:** numbers *not* solutions to algebraic equations **(** π **,** e, and so many more**)**.

Cantor constructed some ingenious ways to organize and "count" all possible solutions to equations having a certain standard form in algebra. These solutions belong to the classification "algebraic numbers". Much of this is beyond what we need but here are a few algebraic equations where the number n can be found mentally; n represents the number which makes the equation "work". Answers are below.

1) $3 + n = 10$; n = ____ 2) $n - 13 = 14$; n = ____ 3) $n^2 + 1 = 10$; n = ____

4) $n^2 + 1 = 11$; n = ____ (Tricky)

The solution to Exercise 3 is 3, not 9, because $3^2 + 1 = 10$.

Exercise 4 requires that n^2 be equal to 10. Read on.

5) The best *approximation* to a solution for $n^2 + 1 = 11$ is (Pick one)**:** 2, 3, 3.1, 3.9____

6) T/F: $\sqrt{10}$ is a correct solution to exercise 4. _____

Answers 1 – 6

1) 7 **2**) 27 **3**) 3 $(-3)^2 + 1$ also works but we are excluding negative numbers for this discussion.

4) See answer to exercise 6. **5)** 3.1 **6)** T

INTO THE ABYSS (THE UNCOUNTABLE) ***Unit 36

In the chart on page 2, we have the classifications of all numbers on the number line, the so-called **real numbers.** Understand and use the chart; memorizing is not needed.

The arrows show that all integers are rational, which in turn are algebraic. All of these algebraic numbers *and* the transcendental numbers make up the real numbers.

The headings "Transcendental" and "Algebraic" gather together all the previous numbers to form the *real numbers*. The transcendentals, such as π and e, are real numbers, but do not contain any algebraic numbers. Neither do the algebraic numbers include any transcendental numbers. The transcendentals stand apart from other numbers, but are in the set of reals.

We have been surprised several times. There are no more integers than natural numbers, no more whole numbers than evens, and no more fractions than counting numbers. The algebraic numbers include not only those sets of numbers just mentioned, but also many *irrationals* (unending and non-repeating decimals). The cardinal numbers of each set, and of all the sets taken together (the algebraic numbers) still do not exceed $\aleph_0$.

True or false: Use the chart on the previous page when needed. The chart and the questions below are designed to help you understand the classifications. There is no need to memorize the chart. Check your answers thoughtfully.

7) An integer is a real number.____

8) Every integer is a rational number ____.

9) Any rational number can be expressed as a ratio of two integers ____

10) A repeating decimal is an irrational number. ____

11) The algebraic numbers include all rationals. ____

12) Some algebraic numbers, but not all, are irrational (not rational). ____

13) Zero is a real number. ____

14) We have not yet discussed the cardinal number of transcendentals, but so far as we know at this point, they are all irrational .____

15) $3\frac{1}{7}$ is irrational.____ 16) π is rational.____ 17) $\sqrt{2}$ is rational.____

18) $.\overline{127}$ is irrational.____ 19) .127…(no pattern indicated) is irrational ____

20) The cardinal number of the set of repeating decimals exceeds the cardinal number of the set of natural numbers.____

INTO THE ABYSS (THE UNCOUNTABLE) ***Unit 36

21) The real numbers include the algebraic numbers and the transcendental numbers.____

22) If a number is rational, it is real, but not transcendental.____

23) Some irrationals are transcendental and some are algebraic ____

24) If the reals are uncountable and the algebraics are countable, then the transcendentals are uncountable. ____

Answers 7 – 24

7) T **8)** T **9)** T, 6 = 6.0000 . . . (Repeating zeros) **10)** F: $.\overline{18} = \frac{18}{99} = \frac{2}{11}$

11) T **12)** T: $n^2+1=10$; $n=3$ (rational) but $n^2+1=11$; $n = \sqrt{10}$ (not rational)

13) T **14)** T **15)** F **16)** F **17)** F **18)** F **19)** T

20) F **21)** T **22)** T **23)** T ! **24)** T !!

We have said that the set of real numbers is the set of *all* numbers on the number line. How do we even *start* to find out how many there are? What questions do we ask? Is there really a way to find out?

Before we look at Cantor's most astounding work we need to understand a kind of proof called an indirect proof, sometimes called a negative proof. It is not difficult but needs some thought and probably more than one reading.

To prove that zero has no reciprocal**:**

Notice that the *opposite* of what we want to prove is **assumed true** at the beginning.

Assume that 0 has a reciprocal, even though no one has ever found it.

Let **R** represent that number which we assumed to be the reciprocal of 0.

Then **R** x 0 = 1, because *any* number multiplied by its reciprocal = 1. (Definition)

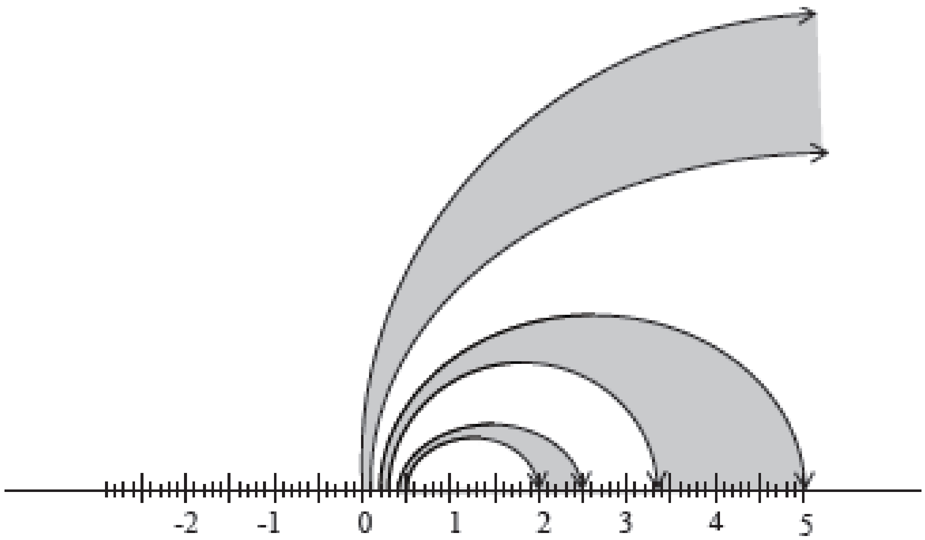
But **R** x 0 = 1 is false, because any number x 0 = 0.

Therefore, the original assumption that **0 has a reciprocal** is false, because that assumption led to a falsehood.

A *consequence* of the beginning assumption is seen in the third step. That consequence is noted as false in the fourth step. The final step is the inescapable conclusion.

INTO THE ABYSS (THE UNCOUNTABLE) ***Unit 36

We use indirect proof in every day conversation: Harriet said as her father walked in from the kitchen, “Dad, I just saw Michael talking with the new girl next door. He didn’t waste any time!” “Oh,” said Dad. “Then the person I have been talking to in the kitchen about next year’s college choices must not be my son Michael.” Dad’s gentle sarcasm made it clear to Harriet that her assumption was wrong. It was not her brother Michael talking to the girl. She went back to the window quietly to get another look at the new boy.



25) Here is a Crazy Six from Unit 29. If you draw an arrow from 9/10 to its reciprocal, it will land on the point corresponding to which of these: 1.1 or 1.1111. . . ___________

26) The fraction that represents the answer to ex. 25 is ______

27) Any decimal between 0 and 1 has a reciprocal greater than _____.

We have raised questions about the cardinal number of the real numbers, all the numbers on the number line. The rational numbers include the integers and the *ratio* of the integers, hence the name “rational”. Yet, there are no more rationals than integers. Cantor listed the rationals in an array (rows and columns) as in unit 29 and this permitted counting them.

Can we do something like that with the real numbers? Can we *list* them to permit counting by 1 to 1 matching with the natural numbers?

INTO THE ABYSS (THE UNCOUNTABLE) ***Unit 36

Let's try. We will concern ourselves only with the decimals less than one because we know that the whole number parts are countable.

1 ⟶ **.**0000000000000000000000000000000000000000000000000000000000001

This is not the smallest decimal we could use to get started. What *is* the smallest? We could insert another zero to make the number one tenth of what it was. But we could repeat this as many times as we please, no matter how small the number gets.

1	.1 0 1 2 3 3. . .
2	.2 0 2 3 4 4. . .
3	.3 0 3 3 5 5. . .
4	.4 0 4 3 6 6. . .
5	.5 0 5 3 7 7. . .
6	.6 0 6 3 6 9. . .
	. . . .

Let's try listing and counting as we go, without worrying about having a smallest possible beginning.

We can see somewhat of a pattern here in matching real decimals with counting numbers. But to continue writing in such a manner would soon confound the writer, and we know that failure is not proof of impossibility.

To do this we ***assume*** that the reals *can* be matched 1 to 1 with the counting numbers, and then consider the consequences that we meet. This might remind you of the proof on page four that zero cannot have a reciprocal. We assumed that 0 does have a reciprocal and found conflict.

But first we need to see clearly yet another example of Cantor's ingenuity. These exercises will allow you to get comfortable with what Canter has given us.

.5 7 6 5 6	Using the array (rows and columns) at left, answer yes/no:
.5 7 6 7 6	28) Is the first number (row) the same as the second number?______
.7 7 6 6 7	29) Is the first number (row) the same as any other number in the array?____
.5 7 6 5 5	30) Are *any* two numbers in the array the same?______
.6 5 7 5 6	*31) Just below .65756, using only the digits 5, 6 and 7, write a five-digit
__ __ __ __ __	number different from each and every five-digit number above it. A *digit*

may be the same as that above it, but a complete number may not.

Answers 25 – 31

25) 1.1111… **26)** 10/9 or 1 1/9 **27)** 1.

28) This is very easy. Just run your eyes across the top two numbers and you know they are not the same.

29) Requires more eye movement and you somehow checked the first with each of the third, fourth and fifth to find that none was the same.

30) "Are *any* two numbers the same?" is a very different story, especially if you had not already done the first two questions. No two numbers are the same.

31) Almost anything you did will help you appreciate what comes next . . .

.5 7 6 5 6
.5 7 6 7 6
.7 7 6 6 7
.5 7 6 5 5
.6 5 7 5 6
— — — — —

Using the same array as in the previous page, we have drawn a **diagonal.** Create a number below the array on the blanks, using only 5, 6, and 7, by having the first digit of the first blank different from the first digit on the diagonal, the second blank different from second diagonal digit, the third blank . . . etc., to the end.

32) The new number created is ______________, perhaps different from the new digit you created above in exercise 31.

Notice that the diagonal, which goes across *and* down, contacts the second number at its second digit's place, the third number at its third digit's place, etc. You could now compose other new numbers in the blanks below which would be different from the top number by selecting different replacement digits, but we won't do that now.

Cantor saw this problem as having an *infinite* number of rows and columns. Even then he found a way to compose a **number that is different** from any and all of the numbers already present, with each of the infinite number of rows being an infinitely long decimal. Now you are beginning to see what Cantor's diagonalization procedure accomplishes. We should remember at this point what prompted all of this investigating:

"Are the real numbers countable? That is, can they be listed to permit 1 to 1 matching with the counting numbers?"

INTO THE ABYSS (THE UNCOUNTABLE) ***Unit 36

Cantor showed that if anyone claimed to have a list, or to be able to *make* a list, to *count all* the members of the set of decimals, he could easily compose a decimal number *not* in their list and thereby show the claim to be false.

That is the heart of Cantor's proof. You now have some idea of how he shows it.

There is a different helpful way to create a new number not in the array. Simply add 1 to the digit contacted by the diagonal, and use the resulting digit for that place in the new number. (If you contact a 9, simply use 0 in the new number).

As before, we use only the decimal parts because we know that the whole numbered parts are countable. Also we know that for two numbers to differ, they need to differ only in one digit.

```
.2 5 0 8 4 0 0 8 9 8 3 4 4 9....
.7 8 6 8 3 2 4 5 2 3 4 8 3 0....
.0 0 0 5 6 8 9 5 3 6 8 5 3 6....
.4 4 0 9 7 0 3 7 3 0 0 9 3 5....
.4 4 3 7 8 9 8 2 4 4 8 7 2 8....
.0 3 4 9 0 2 3 6 4 4 5 5 3 5....
.7 5 8 9 7 4 5 7 6 7 9 0 7 3....
.9 0 0 2 5 8 8 2 7 6 8 9 5 8....
.3 5 1 0 0 9 5 5 5 3 0 6 3 5....
.1 1 6 0 4 4 8 3 3 5 3 7 6 5....
.0 0 9 3 5 6 4 5 8 7 3 0 2 2....
.7 5 9 8 8 7 0 9 8 7 5 9 8 4....
.8 9 8 7 4 5 6 9 0 6 4 9 0 7....
.6 4 8 0 5 0 0 6 9 7 0 4 7 6....
.3. 9. 1. 0. . . . . . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
```

To compose a new number not in this infinite set of numbers, we can start by making the new number's first digit 3. This is all we need to guarantee that in each *first digit*, the new number and the first number will be different.

33) To determine the second digit of the new number we simply make it different from the second digit of the __________ number. To make it different, just add 1. The result is ___ . If we encounter the digit 9 as we do in the 4th number, merely substitute zero. This guarantees that the new number will be different from the third number, at least in their third digits.

INTO THE ABYSS (THE UNCOUNTABLE) ***Unit 36

34) You can see where we are going with this. Above, at the "*bottom*" of the visible part of the array, write the rest of the fourteen-digit visible part of the new number. Change each digit by adding 1 (but change it to 0 if you contact a 9). Trace down the diagonal to the end of the visible part of the infinite array. Not surprisingly, this procedure is called "Diagonalizing" or "Diagonalization".

Answers 32 – 34

32) Since you are selecting digits for your answer, there are 32 possible. (The answer being the same number as the example is coincidence!)

33) second, (or old) **34)** **3 9 1 0** 9 3 6 3 6 6 4 0 1 7

What do you think of all this? Not really difficult? (No written answer needed.)

Think of what has happened! You have written a new number, different from any of the numbers present. And it is easy to see that this diagonalizing could be used for *any* infinite list of numbers where each number has an infinite number of digits, and the procedure could be repeated (but *need* not *actually* be repeated) an infinite number of times, that is, endlessly.

In other words, *anyone's claim* (assumption) to have a plan or pattern for a list of numbers containing *all* of the real numbers is shown to be false, because you have shown that a number different from any number in such a list can always be added to the list. This is the contradiction to the claim. Thus, the assumption is proved incorrect.

This is indeed a dramatic happening in mathematics. Cantor had earlier demonstrated that *each classification* of numbers, from the natural numbers to the algebraic numbers, has the cardinal number $\aleph_0$. But he also showed that all of them together, that is, the algebraic numbers, have the self-same $\aleph_0$ to designate *their* cardinal number. We did not look at this in detail.

Cantor already had many doubters to criticize him about his ideas of the infinite. Imagine the outcry over the announcement that we now have *another* infinity. The so-called real numbers have a cardinal number *greater* than any or all of the classifications it includes.

Where will this end?

INTO THE ABYSS (THE UNCOUNTABLE) ***Unit 36

We now look into the fascinating question about the very large jump from the algebraic numbers, $\aleph_0$**,** to the real numbers, which Cantor called **C.** (Note: We will use **C** as the name of the set as well as its cardinal number.) That jump came up in Exercise 24 on page 4 of this unit. Now we can affirm that the transcendentals *must* be *uncountable* to explain the difference between the number of countable algebraic numbers and the **un**countable reals.

The diagonalizing which creates a *new* number can be repeated countless times. This certainly makes it believable that the jump from the algebraic numbers to the reals is large.

This **C** does not suggest countable but rather *continuum* (pronounced continue-um), a name often used for the real numbers. This emphasizes the very tight, *continuous* packing of real numbers or points on a line, much more densely packed even than the rational numbers. The rational numbers themselves can always have another rational number packed between any two of them by, for example, finding their average.

In Unit 13, you did some work on power sets and saw that a set with 3 members had 8 sets in its power set. You computed that a set with 5 members had 32 sets in its power set. Remember, power set means all possible subsets of a set, including the set itself and $\varnothing$.

35) How many sets are in the power set of a set having 10 members? You don't recall? Here are all the numbers needed to solve the problem, including the answer. Set up the numbers with one of the numbers as an exponent, another as its base (the number which is raised to the power), and the answer**:** 10, 1024, 2. ___________

36) Use your calculator to find the number of sets in the power set of our alphabet. _____________

Answers 35 – 36

35) $2^{10} = 1024$ **36)** $2^{26} = 67{,}108{,}864$

37) The expression to compute the cardinal number of the power set of a set having **n** elements is 2^{n} ; in a set having $\aleph_0$ elements, the cardinal number of its power set is _________.

You have done what was expected if your answer to exercise 37 was $2^{\aleph_0}$. You used $\aleph_0$ as an exponent and 2 as a base which is similar to what you did in previous finite examples. But now you are working with mathematics of the <u>infinite</u>. An amazement like $\aleph_0 + \aleph_0 = \aleph_0$ in Unit 12 should be enough to bring caution about a more advanced expression like $\mathbf{2^{\aleph_0}}$. We do not know that an infinite exponent will behave as it does in finite mathematics.

In the answers above, $2^{10} = 1024$ and $2^{26} = 67{,}108{,}87$, the exponent ^{26} is about 2 ½ times the exponent ^{10}, but the resulting 67,108,862 is many times the result 1,024. As the exponent increases, the result of raising 2 to that power increases "exponentially"; actually, 65,536 times in this case. You could check it.

You might have heard the phrase: "increases exponentially". It is sometimes used to mean roughly that an increase is extremely rapid.

Use your calculator as you need it for these:

38) 2^2 = ____ 39) 2^4 = ____ 40) 2^8 = __________ 41) 2^{16} = ________________

42) 2^{32} = __________________________

43) As we *double* the exponent the result becomes the ___________of the previous answer.

As an effort to feel some sense of the size of $\mathbf{2^{\aleph_0}}$ we can refer back to Unit 32 where we saw written out the value of 2^{800}. It was:

666801443287985427407985179072125779714475832231590816039625781176403723
781763207152143220087155429074292991059343324044588880165411936508036335
605233083004609515757951401455846307828591181402472896501613588660198169
074803747646129116387737 6 exactly.

This number above is not infinite. It is simply very large.
The number below is very, *very* large but still not infinite.

666801443287985427407985179072125779714475832231590816039625781176403723
781763207152143220087155429074292991059343324044588880165411936508036335 ← exponent
605233083004609515757951401455846307828591181402472896501613588660198169
074803747646129116387737 6
2

44) The above number can also be written as: (Fill in the dashed line): $2^{(2^{\text{-------}})}$

INTO THE ABYSS (THE UNCOUNTABLE) ***Unit 36

Answers 38 – 44

38) 4 **39)** 16 **40)** 256 **41)** 65,536 **42)** 4,294,967,296 **43)** square **44)** ^{800}

Exercise 42 is the number of elements in the power set of the power set in ex. 40. That is a very big jump. But we could make much bigger jumps. The answer to ex. 42, over 4 billion, could be raised to the 4 billionth power, and we could repeat that process 4,000,000,000 times. No matter how clever we are in combining finite numbers to produce astronomical results, we will not reach $\aleph_0$. Amir Aczel tells us in "The Mystery of the Aleph", page 219, that $\aleph_0$ enjoys that "inaccessibility" quality (See the first item on our Book List).

We still have not confirmed that $2^{\aleph_0}$ is a correct expression for the cardinal number of the set of real numbers. Fortunately, Cantor left us a proof that the above statement is true. He was able to prove that the *power set of the natural numbers* had the same cardinal number as *the set of real numbers*.

That is, $2^{\aleph_0} = C$

In fact, he proved that the power set of **any** set having **N** members, including any *finite or infinite set*, contained 2^N members. The importance of this proof is shown by its being called "Cantor's Theorem" from then until now, a period of well over 100 years.

We will not explore this remarkable proof but, if you have understood to a reasonable extent the arguments presented to this point, you are probably equipped, and are cheerfully urged, to do the following**:** Upon completing this unit, if your curiosity prompts you, download free from the internet: Peter Suber/Infinite Sets.

Cantor's Theorem is there as Theorem 4, extremely well laid out but not easy, along with other material of real interest. Cantor's Theorem provides the freedom to go on specifying power set after power set, without end! This might allow us to name successive sets as:

$$\aleph_0, \aleph_1, \aleph_2, \aleph_3 \ldots$$

But Cantor did not so name the sets. Why not? Cantor knew that this sequence of Alephs would be sure to attract scholarly minds **only if** it could be appreciated as having a *beginning of* *consecutive infinite sets.* This means that the natural numbers would be followed immediately by the real numbers, and thereby provide a promising basis for further study.

A disturbing development would be the discovery of *another* infinity *between* $\aleph_0$ and **C.** This may not have seemed likely but it explains why Cantor named the cardinal number of the continuum (the real numbers) **C,** rather than $\aleph_1$. He had already proved that the set of natural numbers was the *first* transfinite set, $\aleph_0$, but not that the second was $2^{\aleph_0}$.

So, how could he give the name $\aleph_1$ to the real numbers if he did not *know* that there was not an infinity between $\aleph_0$ and **C?** There are few questions in mathematics any more compelling and significant then this!

The anticipated proof that there was no such interfering infinity was called the "Continuum Hypothesis". Amir Azcel, on page 153 of The Mystery of the Aleph, tells us that "This hypothesis about the orders of infinity is arguably one of the most important statements in all of mathematics". Azcel's book containing this statement was published as recently as the year 2000, more than a century after Cantor's fascinating and controversial work on the infinite.

The generalized version of the continuum hypothesis can be expressed (hopefully) this way**:**

$$\aleph_0\,, \quad C = 2^{\aleph_0} = \aleph_1\,, \quad 2^{\aleph_1} = \aleph_2\,, \quad 2^{\aleph_2} = \aleph_3\,, \quad . \ . \ .$$

You might think, as Cantor thought, that proving the Continuum Hypothesis would not be difficult for him. He is considered to be the creator of set theory and single-handedly had organized the building of the **transfinite** numbers. The process known as diagonalizing was an original method and his persistent use of one-to-one correspondence tied together much of his work. Understanding this gives a look into his genuinely creative mathematics.

But, optimistic efforts by Cantor to prove his Continuum Hypothesis gave way to increasing despair as repeated failures plagued him. Worse, death took his wife and his oldest son in the same year. Cantor himself died in 1918.

`In 1938 the great mathematician Kurt Gödel proved that the continuum hypothesis could not be *disproved* using customary basic rules of set theory. Twenty-five years later, in 1963, the American mathematician Paul Cohen proved that the continuum hypothesis could not be *proved* using those same rules. These two proofs together mean that you can build systems with or without the continuum hypothesis. Most mathematicians working in the field today identify their work as assuming the truth of the continuum hypothesis. (See the final paragraphs of Peter Suber's paper, cited below.)

Cantor's legacies are a highly original theory of transfinites and a firmly grounded theory of sets. Some mathematicians today write of rich opportunities for new study and development in the infinite, or as Cantor named them, transfinite numbers. Few mathematicians have, mostly unaided, created so much that has been so controversial but has also stood the test of time for more than 100 years.

A truly excellent book on Cantor's life and work is Amir Aczel's the *Mystery of the Aleph*. Amazon.com has it. This book is a clear and pleasant narrative with mathematical ideas expressed by a brilliant writer who is considerate of nonprofessional readers.

Peter Suber's <Infinite Sets> is a superb web document for crystal clear and incomparably concise development of Georg Cantor's work. Also, in an adjacent article, Peter Suber has an interesting short work titled "Infinite Reflections".

Among the startling proofs by Cantor shown by Peter Suber are:

a) The number of points on a finite line segment is the same as the number of points on an infinite line.

b) The number of points in a square is the same as the number of points on one of its sides.

c) The number of points in a cube is the same as the number of points on its edges.

If you think that you might benefit in the future from a permanent copy of Peter Suber's "Infinite Sets", type the two words into your internet browser and print out the download.

RECOLLECTIONS AND EXTENSIONS III *Unit 37

1) $10^3 \times 4^2 = 16 \times 10^{\boxed{}}$ 2) 110111_2 =_____ (base ten)

3) A negative 5 and a negative 6 jumper both start at 20. What is the first negative number that they **both** hit? ______

4) What is the LCM (lowest common multiple) of a) 5, 20, 30 ?_____ b) 10^3, 10^5 ? _____

5) The LCM of: a) 50, 10^3, 10^4 ? _________ b) LCM of 25, 10^{10}, 10^{15} = $25 \times 10^{-----}$

6) 28, 35, 42, are called 7n numbers. (Remember?).

a) What do we call 29, 36, 43 . . . .? ____ + ____ numbers.

b) What are the next 3 such numbers after 43? ___________

*c) What are the next 3 numbers after 60, 68, 76, ... ___________________

d) The numbers in c) are called___ n + ___ numbers. (Hint: They differ by 8.)

7) $8.473 \times 10^{270} \div 10^{262}$ = ___________ (No exponents in answer)

8) R means reciprocal of and O_p means opposite of (additive)

a) R 3/4 = ____ b) O_p 3/4 = ____ c) What is the multiplicative opposite of 0? ____

d) R (1 3/4) = ____ e) 98×49^{-1} = _____ f) $8^2 \times \boxed{} = 8^{-3}$ g) $\frac{1}{3^4} = 3^{---}$

h) $2 \div 2^{-1}$ = ___ x ___ = ____ (Don't forget opposite number, opposite operation).

i) What is the additive opposite of O_p (-5)? _____

*9) T/F(Most) Set A = {{3,4,5}, 5, 6} Set B = {3, 4, 5}

a) $5 \in A$ ___ b) $5 \in B$ ___ c) $\{3, 4. 5\} \subseteq A$ ___ *d) $\{\{3, 4, 5\}\} \subseteq A$ ___ e) $B \in A$ ___

f) How many subsets has Set A? _____ g) How many subsets has Set B? _____

10) How many sets in the power set of {&, *, $, #, @, ~, +,}____ (Don't count commas)

Organize work, but leave out work that you can do mentally and with confidence.

11a) 6 x 5 - 3 = ____ b) 16 - 3 x 5 = ____ c) 6(8 - 2) + 4(9 + 9) =

d) $3\sqrt{36} + 3(3^3 - 20) + 3(3^3 - 4 \times 5)$ = e) $\left(\sqrt{29}\right)^2 - \left(\sqrt{25}\right)^2$ =

RECOLLECTIONS AND EXTENSIONS III — *Unit 37

100 terms

501 to 510 is 10 terms. 500 to 510 is 11 terms

12) Find the sums of the series: a) 101+ 102 + 103 + . . . 199 + 200_______

(Be careful) b) 100 + 101 + 102 + . . . 199 + 200 _______

c) What must be done to the series in a) to equal the series in b)?________________

13) How many terms has the series: a) 120 + 122 + 124 + . . . + 198 + 200 ?______

*b) 99 + 199 + 299 + . . . + 1399 + 1499 ? ______

*c) 46 1/2 + 48 + 49 1/2 + 51 + . . . + 298 1/2 + 300? _____(Hint: Double each term)

*14) Find the sum of the series 205 + 212 + 219 + . . . + 352 + 359

15) How many degrees between the hands of a clock at

a) 3:00 _____ b) 6:00 _____ c) 2:00 ______ d) 5:00 ______

e) How many degrees in the **reflex** angle of the hands at 7:00? _____

f) Same question as e) for 11:00 ______

16) If a class was asked to find the number of degrees between the hands at 12:30, what, in your opinion, would be the most popular **wrong answer?**

a) _______ Right! If you said 180°, that is the **right wrong answer.**

b) Now, don't give the right wrong answer; give the right *right* answer here: _____ Actually there may be no such thing as a *wrong* right answer(?). So, simply give the right answer. Of course there *is* such a thing as a wrong *wrong* answer. Maybe you got such an answer for 16a). Don't forget to do 16b).

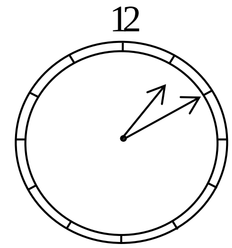
17a) You reasoned, didn't you, that while the minute hand traveled from the 12 down to the 6, that it went half way around the clock? Meanwhile, the tortoise-like hour hand went half way from the 12 to the ____, so it went 1/2 of _____° = _____°.

b) So the measure of the angle between the hands at 12:30 = _____°

RECOLLECTIONS AND EXTENSIONS III

***Unit 37**

**18) In the diagram to the right, the minute hand has passed the hour hand on its way from 1 o'clock to 1:10, the time it now shows.



*a) How many degrees are between the hour hand and minute hand?_____

** b) How many degrees in the obtuse angle between the hands at 9:30? _____

**c) How many degrees in the acute angle at 9:40? _____

Mental Math

19a) 50% of 120 = ____ b) 25% of 120 = _____ c) 12 1/2% of 160 ____

d) 10 x 866 = _____ e) 5 x 866 = ______ f) 50 x 668 = ______

g) $\sqrt{160000}$ = _____ h) 87 1/2% of 160 = ____ i) 20 x 20 = _____

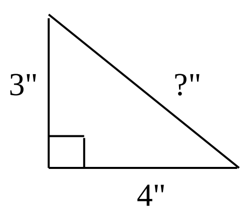
j) 21 x 19 = _____ k) 18 x 22 = _____ l) 30 x 30 = _____

m) 280000 x 32000 = 896 x 10^{----} n) 3900 x 4100000 = _____x 10^{----}

o) 60 - 3/4 = _____ p) 60 – 2 3/4 = _____ q) 120 – 4 3/5 = _____

r) 457 + 299 =_____

20)



_______"

Use the rule of Pythagoras to find the hypotenuse in the easiest Pythagoras problem you will ever get.

**21) Brianna dropped her favorite tennis ball out of her 3rd story apartment window in New York City. The height was 32 feet. Brianna's ball bounces half as high as it drops when it hits a concrete New York City sidewalk as it was doing now. What total vertical distance (both up and down) does Brianna's ball travel before coming to rest. Ignore influences like sidewalk cracks, gum on the sidewalk, playful dogs, etc. ___________.

RECOLLECTIONS AND EXTENSIONS III *Unit 37

See discussion below:

Your analysis of Brianna's bouncing ball problem could refer to the similarity of key ideas in Halfwalk, Unit 15. At the same time, recognize the differences.

The series of bounces is very much like the "stages" in Halfwalk, but the first bounce in Brianna's problem is <u>dropped</u> from above, giving only 1 occurrence of 32 feet. Each subsequent bounce has 2 equal distances, up and down.

Also, when the terms of the series (bounces) drop to less than one, you are confronted with a familiar infinite convergent series. It might help to draw a diagram.

Answers

1) ^{3} **2)** 55 **3)** -10 **4a)** 60 **b)** 10^5

5a) 50,000 or 5×10^4 **5b)** ^{15}

6a) 7n + 1 **b)** 50, 57, 64 **c)** 84, 92, 100 **d)** 8n + 4

7) $= 8.473 \times 10^8 = 847{,}300{,}000$

8a) 4/3 or 1 1/3 **b)** -3/4 **c)** There is no multiplicative opposite of 0. **d)** 4/7

e) 2 **f)** 8^{-5} **g)** $^{-4}$ **h)** 2 x 2 = 4 **i)** -5

9a) T **b)** T **c)** F **d)** T **e)** T **f)** 8 **g)** 8

10) $2^7 = 128$ **11a)** 27 **b)** 1 **c)**108 **d)** 60 **e)** 4

12a) 15050 **b)** 15150 **c)** Add 100

13a) 41 **b)** 15 **c)** 170

14) 6486

15a) 90° **b)** 180° **c)** 60° **d)** 150° **e)** 210° **f)** 330°

16a) 180° **b)** 165° **17a)** 1 - - 30° = 15° **b)** 165°

****18a)** 25° **b)** 105° **c)** 50°

19a) 60 **b)** 30 **c)** 20 **d)** 8660 **e)** 4330 **f)** 33,400

g) 400 **h)** 140 **i)** 400 **j)** 399 **k)** 396 **l)** 900

m) ^{7} **n)** 1599 - -^{7} **o)** 59 1/4 **p)** 57 1/4 **q)** 115 2/5 **r)** 756

20) 5 ft. **21)** 96 ft.

No quiz for Unit 37.

RUSSELL'S PARADOX ***Unit 38

Warning! This unit is not for the faint of heart.

If the statement "I am lying" is true, then it is false, and if false, then it is true.
(This is not Russell's Paradox)

If you can understand to some extent the elusive statement above, then you are probably ready for this paper. Bertrand Russell was one the greatest philosophers of the 20th century. Very early in the century he and his mathematician friend Alfred North Whitehead were well into the work of trying to put all of mathematics on a completely logical footing. This happened to be about the time that Georg Cantor's mathematical work was suffering due to his illness. Bertrand Russell was not afflicted as Cantor was, but, like Cantor, he had run into a daunting problem which simply would not yield. Russell spent the summer evenings of the two years 1903 and 1904 walking, discussing and pondering a problem about – what else? – sets. What disturbed Russell most was that the problem was in the very foundations of mathematics and logic, exactly where his and Whitehead's work was focused.

There is another popular paradox below which more nearly parallels Russell's problem than the one given above in small print. It is known as the Barber Paradox.

In a certain town there is only one barber and he shaves all men (and only those) who do not shave themselves. If we have two lists, the first being those who shave themselves and the second those whom the barber shaves, in which list is the barber?

Note: An answer is provided on page 2. Peek at it if you cannot seem to get started and at other times when you need help. Use your own words.

1) Give a two part answer to the barber question by considering his inclusion in each list one at a time and do your best to state why he does or does not belong there**:**____

____

____

____

____

____

____

RUSSELL'S PARADOX

*****Unit 38**

Answer

Your answer will not be exactly like the one given, but it should clearly point out the contradiction in each case, even if you don't use the word contradiction.

If a man is put on the list of those that do not shave themselves, then he must be shaved by the barber, which means that if the man is the barber, he shaves himself. This is a contradiction because it says that if the barber does not shave himself, then he shaves himself.

On the other hand, if a man is put on the list of those that shave themselves, when the man is the barber then the barber shaves himself. But this is a contradiction because it says that if the barber shaves himself, he does not shave himself (he shaves only those who do not shave themselves).

Russell constructed his paradox using pure sets rather than barbers or liars, etc. Pure sets {except $\varnothing$} have members that are not specified as anything in particular. This generalizes the problem, that is, makes it work for sets of *any* things, including sets of ideas. Russell divided sets into two kinds: a) sets that are not members of themselves, and b) sets that are members of themselves. As you found in Unit 13 "Sets, Subsets, Power Sets and Upsets", there is a difference between being a member of a set and being a subset of a set:

$$\text{If the set } A = \{4, 5, 6\}, \text{ then } 5 \in A \text{ and } \{5\} \subset A.$$

A set of ordinary objects such as a set of pencils, is not itself a pencil, so a set of pencils is not a member of itself. (It is a subset of itself.)

An example of the other kind of set, that which *is* a member of itself, is the *set of ideas expressed in this unit*. The *set of ideas expressed in this unit* <u>is</u> <u>itself</u> one of the ideas expressed in this unit. We just now expressed it (ponder this). Another such set is the set of all printed phrases on this page. If we were to have a list titled "The set of all printed phrases on this page", among the phrases in the list would appear "The set of all printed phrases on this page". *That* phrase is certainly a member of itself.

RUSSELL'S PARADOX ***Unit 38

Now, says Bertrand Russell, consider all the sets of the ordinary kind, that is, all the sets which are not members of themselves. All these ordinary sets are gathered (mentally) into one set which we will call set N, which means all sets not members of themselves. We will make two lists, one of N sets and one of M sets, M for Members of themselves.

Members of Set N	**Members of Set M**
•	•
•	•
•	•
Set N?	**Set N?**
•	•
•	•
•	•
•	•

2) Consider membership of set N *in each set* N and then M, and tell clearly why it does or does not belong there. Write your two-part answer.

____

____

____

____

____

____

____

RUSSELL'S PARADOX ***Unit 38

Answer 2

2a) Assume that set N is a member of set N. Its membership in N makes N a member of itself. This quality of being a member of itself makes set N an M set, by definition of M. This contradicts the assumption that N is a member of set N, so N is not an N.

b) Assume that set N is a member of set M. Membership in M means set N is a member of itself, that is, of set N. This contradicts the assumption that N is a member of M, so N is not an M.

These answers may be stated briefly as:

If N is an N then it is not an N; if N is an M, then it is not an M.

Your answer may be quite different in its wording and still be correct. But deeming your own answer correct should come only with mental honesty, well applied.

Russell's paradox is very elusive. If you reach a point where you think you understand it you might find that it slips away easily. If you are persistent you might get it quite firmly in mind. This is not a small accomplishment. The next time you have to wait for someone, or for some other reason you have time on your hands, try going over it mentally. You might surprise yourself. There is also the chance that going over it and over it will result in memorizing without full understanding. But even this can be a help in finally arriving at a satisfying sense of right reasoning which does not "solve" the paradox but instead leads more *clearly* to a contradiction.

A good paradox is not easy to explain away. Russell wrote to the German logician, Gottlob Frege, who was then getting ready to publish his own work on the foundations of logic. Frege publically accepted that Russell's paradox undermined his own years of work. While mathematicians and logicians continue to argue about paradoxes, Frege's admission of the failure of his work deserves our respect, even today. (See Barrow's *Pi in the Sky*, page 111, our book list #3.)

The four-volume set "The World of Mathematics" edited by James R. Newman, on pages 1674-1675, our book list #13, has a fine exposition of Russell's paradox and intensely interesting allied topics (not easy!) in surrounding pages.

TO END OR NOT TO END? IS THAT THE QUESTION? **Unit 39

The sum of a set of numbers such as 1 + 3 + 5 +7 + 9 + . . . is called a **series.**

A set of ordered numbers such as 1, 2, 3, . . . is called a **sequence.**

The first is an infinite series and the second an infinite sequence.

When the sum of an infinite series, as in the unit *Halfwalk*, 1/2 + 1/4 + 1/8 . . . is getting closer and closer to a particular limiting number, we call this series *convergent.* An infinite series like 1 + 3 + 5 . . . which has no limiting sum is *divergent.*

In the 14th century a French Priest and mathematician named Nicole Oresme was one of a group aware of divergent and convergent series. Even at that time, one of the series quite well known was the *harmonic series*. $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+$ . . . It is the sum of the reciprocals of the counting numbers. Some argument centered around whether this **harmonic** series is convergent or divergent, but the prevailing belief among mathematicians was that the series was convergent. This was because (like Halfwalk) the harmonic adds smaller and smaller numbers which would eventually choke off.

Possibly, Oresme's genius told him that a solution to the problem of proving the harmonic series either convergent or divergent could involve Halfwalk in *some* way, although he would not have used that name for the series. That is a private name for these pages only.

Calvin Clawson, on page 61 of his book *Mathematical Mysteries*, our book list #3, gives us a brief but convincing account of Oresme's encounter with the harmonic series which we expand below.

Now comes a favorite tactic which mathematicians sometimes use in proofs. They change the question to be answered into a different question for which the outcome will be the *same but more easily understood.* .

It is like being asked to add 4986 + 5014.

1) Instead, you decide to add 5000 + 5000. The answer is_________.

Easily changed question (if you think of doing it!), same answer. But the change permits mental computing.

TO END OR NOT TO END? IS THAT THE QUESTION? **Unit 39

Oresme's changes were different but very effective and marked his genius. In the first change, **A** (below), Oresme grouped terms of the harmonic series into the associations shown.. This is perfectly legal since this regrouping grows out of the associative property of addition.

A $$\frac{1}{1}+\frac{1}{2}+\left\{\frac{1}{3}+\frac{1}{4}\right\}+\left\{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right\}+\left\{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right\}+\ldots$$

(You can already see some hints of Halfwalk here)

Oresme's next change made each term as small as the smallest in its group above.

B $$\frac{1}{1}+\frac{1}{2}+\left\{\frac{1}{4}+\frac{1}{4}\right\}+\left\{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right\}+\left\{\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\right\}+\ldots$$

See that the fractions in **B** will add to *less* than will those in **A,** the harmonic series. Be sure none in **B** is larger and some are smaller. **Good**. Then both **A** and **B** can extend as far as we like and **B** will always run smaller.

2) Continuing the pattern in **B:** What are the denominators next after the

a) 16ths?_____ b) how many? ____ And what denominators are next after those? c)_____ d) how many?______

In the second trick (tactic), we saw the denominators in each group changed to the size of **the largest denominator in its group**. Just a minute!! Is *this* legal? And why is it done!? It is fine if we want each fraction in this manufactured series to be smaller than (or *as small as*) its corresponding term in the grouped harmonic series above it, and Oresme did want that. The other question—Why?!—will be answered soon.

A $$\frac{1}{1}+\frac{1}{2}+\left\{\frac{1}{3}+\frac{1}{4}\right\}+\left\{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right\}+\left\{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right\}+\ldots$$

B $$\frac{1}{1}+\frac{1}{2}+\left\{\frac{1}{4}+\frac{1}{4}\right\}+\left\{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right\}+\left\{\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\right\}+\ldots$$

C $$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$$

At this point all questions are basically answered.

3) In **C,** the halves formed from the fractions in **B** are collected. Will that supply of halves ever end? ____.

4) Will the extending of the series in **C** give ever larger sums? ______

5) Therefore the series in **C** is (convergent/divergent) ________________________

TO END OR NOT TO END? IS THAT THE QUESTION? **Unit 39

A $$\frac{1}{1}+\frac{1}{2}+\left\{\frac{1}{3}+\frac{1}{4}\right\}+\left\{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right\}+\left\{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right\}+\ldots$$

B $$\frac{1}{1}+\frac{1}{2}+\left\{\frac{1}{4}+\frac{1}{4}\right\}+\left\{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right\}+\left\{\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\right\}+\ldots$$

C $$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$$

6) **T/F** The series in **B** has the same values as the series in the divergent series C; they are merely written differently ___

7) Therefore, the series in B is (convergent/divergent) ____________________

8) **T/F** The harmonic series in A grows even more quickly than the divergent series in B___.

9) Therefore, the Harmonic Series in A is (convergent/divergent) __________________

QED ("quod erat demonstrandum"; Latin for "that which was to be demonstrated").

At the time, those interested in this proof were highly surprised; many unpleasantly. Prejudice exists in our points of view, and faulty opinions held for a long time can assail one deeply when the plain truth is found to be contrary.

Answers 1 – 9

1) 10,000 **2a)** 32nds **b)** 16 **c)** 64ths **d)** 32 **3)** No **4)** Yes

5) Divergent **6)** T **7)** Divergent **8)** T **9)** Divergent

---

A glance tells you that the **Harmonic series** is in 1 - 1 correspondence with the counting numbers.

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\bullet\bullet\bullet$$
$$1+2+3+4+5+6\bullet\bullet\bullet$$

10) You might think it strange that each of these series has the same number of members. What is that number for the harmonic series? _______

In the **Halfwalk series**, you can match the powers of 2 in the denominators with the counting numbers.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}\bullet\bullet\bullet$$
$$\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}\bullet\bullet\bullet$$
$$\rightarrow 1,\ 2,\ 3,\ 4,\ 5,\ 6\bullet\bullet\bullet$$

11) The comment to the left tells you the number of terms in the Halfwalk series. It is: ______

To End or Not to End? Is That the Question? **Unit 39

"To End or Not to End" applies to the alleged unending of two basic kinds of series: convergent and divergent. Common sense can bring confused thinking about the infinite. Of course our common sense is confused. Our sense of the infinite is not at all common. You might ask yourself: "Which is easier to visualize, an infinite convergent series like Halfwalk, or an infinite divergent series like the whole numbers?" Remember that each has the cardinal number $\aleph_0$ **(Aleph null),** the same as the cardinal number of the counting numbers, so *neither* kind of series ends. Neither convergent nor divergent series are visualized easily simply because they are infinite.

Answers 10 – 11

10) $\aleph_0$ 11) $\aleph_0$

This topic is in *Mathematical Mysteries, by Calvin Clawson,* #6 on our *Book List.* The Internet has some interesting sites on The Harmonic Series. An entertaining one concerning harmonics in music can be had through Google (or your search engine).

In the search space enter **Harmonic Series**. From the resulting listings select the first one that mentions Sarah Tulga.

Note: We wrote the Halfwalk series sometimes *with* 1 at the beginning and sometimes *without*. It does no harm. $\aleph_0$ is the cardinal number of terms in Halfwalk either way because $\aleph_0 + 1 = \aleph_0$. (See the unit "A Glimpse into the Infinite".)

RUSSIAN PEASANT MULTIPLICATION *Unit 40

The strange method of multiplication described here gets its name from historic use by peasants in many parts of Russia and other countries, and as early as 2000 B. C. in Egypt. Here is what they did:

To multiply 38 x 44

First write the numbers beside each other. Then divide the left-hand number by 2 and keep on dividing the result by 2 down the left side until you get down to a result of 1. Forget remainders!

That is one of the strange things about this method. You just ignore remainders.

Next, double the right-hand number and keep on doubling until you have as many answers as there are in the first column.

38 x 44
19 x 88
9 x 176
4 x 352
2 x 704
1 x 1408

Now we have another strange and important part. Select those numbers in the right-hand column which are opposite *odd* numbers in the left-hand column. In this case, those are the numbers 88, 176 and 1408.

88
176
1408
1672

Add these numbers and you have the answer to the original multiplication example. 38 x 44 = 1672.

Here is another example to illustrate the method: 97 x 52.

97 x 52	
48 x 104	
24 x 208	
12 x 416	52
6 x 832	1664
3 x 1664	3328
1 x 3328	5044

Do these four problems by Russian Peasant Multiplication.

24 x 21   108 x 61   15 x 103   16 x 804

RUSSIAN PEASANT MULTIPLICATION

***Unit 40**

Answers

24 x 21	
12 x 42	
6 x 84	168
3 x 168	336
1 x 336	504 ←

108 x 61	
54 x 122	
27 x 244	244
13 x 488	488
6 x 976	1952
3 x 1952	3904
1 x 3904	6588 ←

15 x 103	103
7 x 206	206
3 x 412	412
1 x 824	824
	1545 ←

16 x 804
8 x 1608
4 x 3216
2 x 6432
1 x 12864 ←

The mystery of *why* this multiplication system works is partly revealed using binary numbers. We will illustrate using the first of the four exercises at the bottom of page 1.

24 x 21 = 504. Change 24 to binary numeration. ___________.

Use 24 expressed in binary form but keep 21 as is.

11000_{two} x 21

→ 8 x 21 = 168

→ 16 x 21 = 336

504

You might feel that this little demonstration does not do much to *explain* the Russian Peasant method. It seems to do little more than confirm that it works. We need to find something more convincing.

Look up at the work at the top right. Since the multiplier 16 is a power of 2, repeatedly dividing by 2 always give a whole number, remainder **0**. In this case, the answer to the original multiplication problem is the final number in the doubles column. Since the final 12,864 is opposite the *odd* number 1, it is the only member of the list of those to be added so it is simpler to leave it where it is. This problem is like many you had in the Mental arithmetic unit. Isn't it *always* true in any Russian Peasant multiplication that the *last* number in the doubles list is opposite the odd number 1 and is therefore a number to be added? _____

The answers to the blanks above are 11000 and "Yes".

We still do not have a full explanation of the "magic" that allows us to drop remainders. So, looking again at the very first illustration you saw, repeated at right, what happens when we drop the remainder 1 from 19 ÷ 2? We cannot commit such a violation of common sense without some kind of consequence. You could say that we are pretending that 19 is 18 so that when we divide by 2 we have remainder 0, not 1.

38 x 44	88
19 x 88	176
9 x 176	1408
4 x 352	1672
2 x 704	
1 x 1408	

RUSSIAN PEASANT MULTIPLICATION *Unit 40

Since, by dropping the remainder, we have decided to multiply only <u>18</u> x 88, not <u>19</u> x 88, *we are dropping <u>one 88</u>*, not merely 1. This violation is rectified by adding one 88 back in.

Likewise, we are treating 9 x 176 as 8 x 176 so we must add in a 176, and we are treating 1 x 1408 as if it were 0 x 1408 so we must add back in the 1408. All this was done above at right where we see 1672 which is the correct final answer.

Now, we hope, you understand more than an ancient peasant. But, for an unschooled person, no knowledge of multiplication is needed so long as he can double large numbers, perhaps by adding the number to the number. The halving might be done by trial and error**:** guess the half, double it, thoughtfully guess again, etc. It is mystifying to speculate how someone might do even this without place value and a numeration system.

FIBONACCI AND FRIENDS **Unit 41

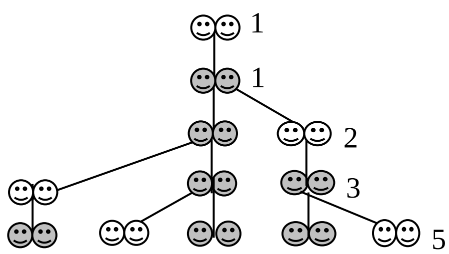


Figure 1

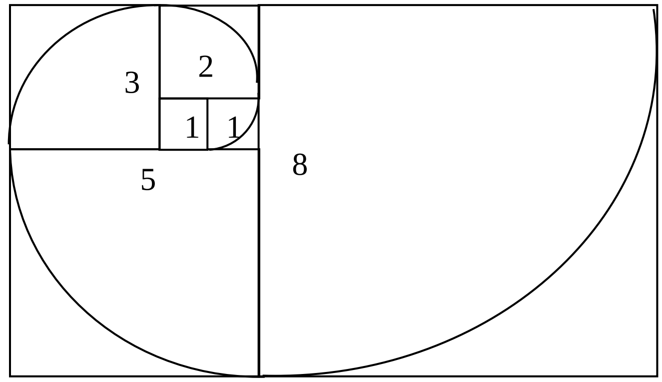


Figure 2

CALCULATOR O.K. when needed.

Figure 1 above is about Fibonacci (sounds like Ribbon Archie) and his rabbits, a well known story among mathematicians and readers of popularized math books. Figure 2 was known by the Greeks at least 1000 years earlier, in very different circumstances. In fact, the main idea clearly common to both diagrams seems to be a set of numbers.

1) What number is needed in Figure 1 to make a clearer sameness with Figure 2? _____

After you have answered exercise 1 (Answer is on page 2), look at the two diagrams and see what number would follow 8. Because the numbers follow the same pattern in the two very different diagrams, we will assume that both will continue in the same way. If so, we can treat the numbers as a pattern to be extended, regardless of any practical matter such as rabbits' birth rate, Figure 1, or the expansion of a spiral, Figure 2. So, continue the sequence for 6 more members:

2) 1, 1, 2, 3, 5, 8,____, _____, ___, _____, _____, _____.

But, the diagram for the expected birthrate of Fibonacci's rabbits is based on the idea that rabbits follow strict rules in their mating habits. Fibonacci used mathematician's license to consider the situation and its quantities perfect in order to apply numbers to them. Leonardo of Pisa, a name which somehow morphed into Fibonacci, must have had a sense of humor to think up this particular idea. The schedule thrust upon the rabbits goes like this: A "new" pair (unshaded in Figure 1) must wait a month until reaching maturity and then mating (shaded in Figure 1). Looking at Figure 1, note that the original pair continues to live and mate, perhaps forever. (Trace them straight down.) In Figure 1, the numbers at the end of each row tells the total number of rabbit pairs alive at that time. No rabbit dies.

FIBONACCI AND FRIENDS **Unit 41

Answers 1 – 2

1) 8 **2)** 13, 21, 34, 55, 89, 144

3) In the chart at right, continue the middle Column of Fibonacci numbers. The first 6 are already done. Do not use the third column yet. The 21st Fibonacci number is 10,946. Work carefully with your calculator. Mistakes come easily in long columns.

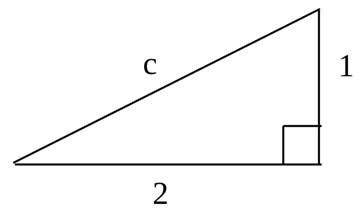
4) In the third column you will write as a decimal the ratio of the number of rabbit pairs in the previous row to the number below it in the present row. This is a tedious business, so after doing a few, skip down to the 17th Fibonacci number and finish from there to the 25th number. Notice what is happening to the ratio as you go.

5) If your calculator has a ten digit readout, at what number do the readouts stabilize; that is, at what number (in the left-most column) does the ratio no longer change?____ If you have a computer calculator you could do the final problem on it and you might be surprised at the "more precise" ratio it gives.

1	1	
2	1	
3	2	
4	3	
5	5	
6	8	
7		
8		
9		
10		
11		
12		
13		
14		
15		
16		
17		
18		
19		
20		
21		
22		
23		
24		
25		

FIBONACCI AND FRIENDS **Unit 41

6)



Use your calculator to find the length of the hypotenuse of the right triangle. Add that to the short leg of the triangle and divide that result by the long leg's length.__________________. Compare that to the final number in the table on the previous page. Pythagoras and Fibonacci seem to have something in common.

1	1	1
2	1	1
3	2	**.5**
4	3	**.6666666667**
5	5	**.6**
6	8	**.625**
7	13	**.615384615**
8	21	**.619047619**
9	34	**.617647059**
10	55	**.618181818**
11	89	**.617977528**
12	144	**.618055556**
13	233	**.618025751**
14	377	**.618037135**
15	610	**.618032787**
16	987	**.618034448**
17	1597	**.618033813**
18	2584	**.618034056**
19	4181	**.618033963**
20	6765	**.618033999**
21	10946	**.618033985**
22	17711	**.61803399**
23	28657	**.618033988**
24	46368	**.618033989**
25	75025	**.618033989**

Answers 3 – 6

Did you notice the oscillation (bouncing) of the results?

3) see table

4) see table

5) 24, or if you prefer, 25.

6) 1.618033989

Exercise 6 gave a surprising answer. Or did it?

FIBONACCI AND FRIENDS **Unit 41

*****7) Suppose you are told that there is one number in the world such that one more than the number is its reciprocal. What is that number correct to 3 decimal places?

Well. How does one go about solving such a problem? Guess? Yes, if you do some thinking about it, too. You could recall Crazy Six and remember that the interval between 0 and 1 contains the reciprocal of every real number greater than 1. So you could start entering three place decimal numbers, press the 1/n key and see if the result is close to one plus the number, adjusting your next entry based on the result. Or, state the problem in the language of math to help visualize it**:**

$$\mathbf{n + 1 = 1/n}$$

But this is very close to requiring algebra, something you were told would not happen. Give it some thought and work, and then peek at the answer on the next page.

Answer**: n = .__ __ __** Reciprocal**: =__.__ __ __**

Round to 3 decimals. You will know you have the correct answer when you press the 1/n key and see a number 1 greater than the number you had in the window.

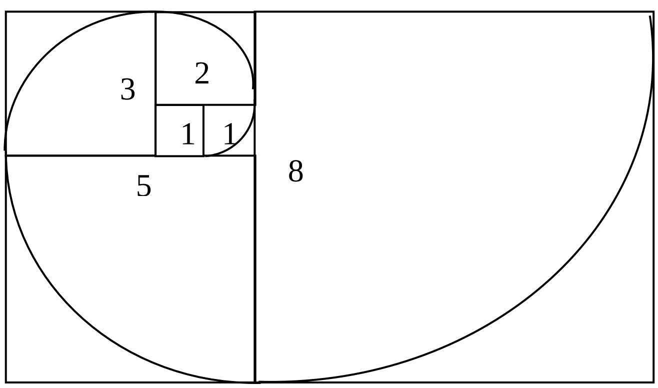


Figure 2

Answer 7

7) Number is **.**618, Reciprocal is 1**.**618

FIBONACCI AND FRIENDS **Unit 41

The ancient Greeks took great interest in Fibonacci's spiral. No doubt you remember that Fibonacci could claim no credit for this because he was yet to be born more than a thousand years later. The Greeks were of strong opinion that this rectangle was very close to the most beautiful that a rectangle could be. The way that the rectangle unfolds itself in Figure 2 assures that subsequent rectangles would continue to be closer and closer to the "divine ratio" of .618. Actually 5/8 = .625, followed by 8/13 = .615 (not shown). The Greeks were serious about this. It appears frequently today, especially in Greek architectural ruins.

Pascal's Triangle

1

1 1

1 2 1

1 ___ 3 1

1 ___ ___ ___ 1

1 5 10 10 5 1

1 6 ___ ___ ___ 6 1

1 7 21 35 ___ ___ 7 1

1 8 ___ 56 ___ 56 28 8 1

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

8) Use patterns above to fill in the missing numbers and then write the next row in Pascal's triangle. There is room enough.

When Blaise Pascal appeared on the mathematical scene quite early in the sixteen hundreds (1623), he was soon recognized as one of the greatest mathematicians living. Pascal recognized that a **triangle**, which probably had been used earlier, would serve his purposes nicely. Much of its richness lay in its help in writing algebraic equations, probability and a richness of patterns in its various rows, columns and diagonals.

FIBONACCI AND FRIENDS **Unit 41

Answer 8

1

1 1

1 2 1

1 **3** 3 1 **13**

21

1 **4** **6** **4** 1 **34**

1 5 10 10 5 1

1 6 **15** **20** **15** 6 1

1 7 21 35 **35** **21** 7 1

1 8 **28** 56 **70** 56 28 8 1

1 **9** **36** **84** **126** **126** **84** **36** **9** **1**

We can see above where Fibonacci is in this triangle. He actually pervades the whole diagram. Put a convenient straight edge (card or ruler, etc., a narrow one is best) horizontally under the pair of tens. Rotate the edge counter-clockwise until the edge lies neatly under the rising line of numerals 1 6 10 4, as shown. Notice that the sum of these four is 21, a Fibonacci number. If you slide the edge, still at the same angle, up or down, you should find sets along the straight edge adding up to a Fibonacci number! (See chart above.)

It seems almost mysterious that the Fibonacci sequence keeps popping up unexpectedly, especially in nature in leaves, flowers, certain pine cones (spirals), and others. Don't miss the colorful spectacles awaiting anyone who will search Google for Fibonacci.

In 1963 there was a magazine established called the Fibonacci Quarterly. This, among many other publications, has a web site with links for easy access from the above site. This is a mine of Fibonacci information for interested students, including those looking for topics for special reports.

FIBONACCI AND FRIENDS **Unit 41

A final example of the work of Leonardo of Pisa (the location of the famous leaning tower) is the use of continued fractions. Be careful. There is an unusual richness of shortcut opportunities but these can let you slip and stumble instead of glide. The secret is concentration.

Try to do each exercise mentally, using a problem or a part of it within another.

Recall that 4 ¼ means 4 + ¼ = 17/4. The reciprocal of 17/4 is $\frac{1}{\frac{17}{4}}$ which is $\frac{4}{17}$.

9a) $\dfrac{1}{1+\frac{1}{4}}$ = ____ b) $\dfrac{1}{1+\frac{1}{3}}$ = ____ c) $\dfrac{1}{1+\frac{1}{5}}$ = ____ d) $\dfrac{1}{1+\frac{2}{3}}$ = ____ e) $\dfrac{1}{1+\frac{3}{5}}$ = ____

f) $\dfrac{1}{1+\frac{5}{8}}$ = ____

Note: Your answers to d), e) and f) should be consecutive members of the Fibonacci ratios.

10) Simplify these continued fractions. Start at the bottom of each and work your way up.

a) $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1/2}}}$ = _____ b) $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1/2}}}}$ = ___________

c) $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1/2}}}}}$ =______________

Note: Each answer to a), b) and c) should contain a member of the Fibonacci ratios.

We have considered Fibonacci from the viewpoints of 7 topics, some of them seeming rather remote from the basic sequence but some, with contemplation, do not.

1. Fibonacci's rabbits
2. The golden rectangle
3. Pythagoras' right triangle with legs 1 and 2.
4. Calculating the ratios of consecutive Fibonacci numbers
5. Finding the only number such that 1 greater than the number is its reciprocal.

6. Pascal's Triangle
7. Continued fractions

There are two books in our book list particularly rich on the topics of Fibonacci and continued fractions. They are references #6, "Mathematical Mysteries" by Calvin Clawson and #14, excursions in Number Theory by C. Ogilvy and J. Anderson.

Answers 9 – 10

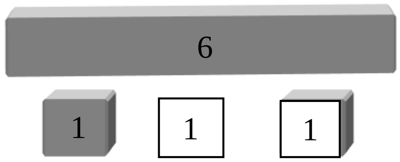
9a) 4/5 **b)** 3/4 **c)** 5/6 **d)** 3/5 **e)** 5/8 **f)** 8/13

10a) 1 5/8 or 13/8 **b)** 1 8/13 or 21/13 **c)** 1 13/21 or 34/21

The remoteness vs**.** nearness to the basic idea of the Fibonacci sequence could be judged partly according to whether or not the present idea reaches back and uses one or more of its previous members or procedures to form the present one. In extending the sequence 1, 1, 2, 3, 5, 8, 13, . . . , we see that 5 + 8 is used to form 13, so we continue to reach back to 8 + 13 to form 21**,** etc. etc. This same idea would certainly apply to the Golden Rectangle, Fibonacci's rabbits and continued fractions, but thoughtful inspection may be needed to see this. The others in the list are more obscure, but perhaps Pascal's Triangle would yield to quiet thought. This act of reaching back and using the results to progress is called recursion. Wikipedia Encyclopedia on the internet confirms that the Fibonacci sequence is indeed recursive. Strictly speaking, however, there are 1 or 2 additional requirements for recursion. Try out Wikipedia. It is like a whole new world but not viewed as altogether authentic because of its voluntary, continuing development.

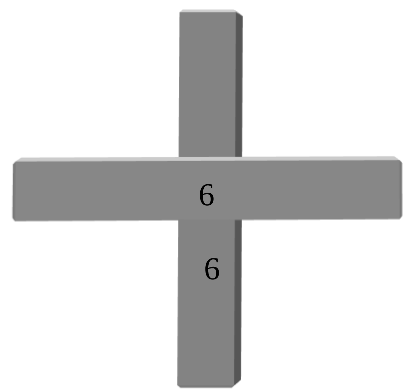
Rods and Stamps

*Unit 42

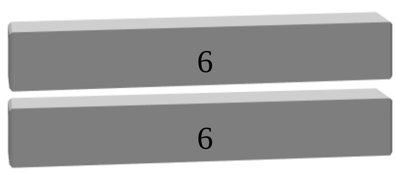


1) There is a rod 6 units long in the above figure. Notice the one unit cubes below it. The middle figure is not a cube; it is a one unit a *stamp*. The third figure is the *stamp* placed on one of the faces of the cube. Consider all the rods and cubes to be suspended in space. They do not rest on any surface. How many times could we carefully place a stamp on the 6 rod, including the hidden base as well as the hidden backs and ends?_____

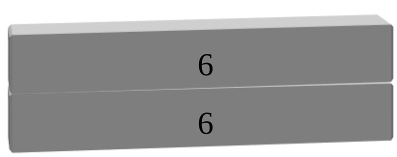
2) Treat the stamping of the figure to the right as you did in exercise 1 but do not include the area where one figure rests against the other._____
A rather likely wrong answer is 51 stamps.



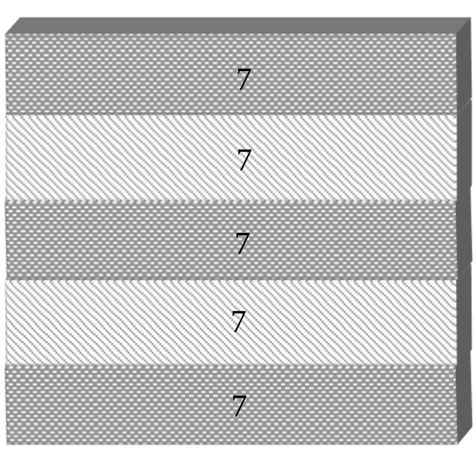
3a) 2 separate logs. How many stamps? _____



b) 2 joined logs. How many stamps? _____



4a) Number of stamps for below if all rods are lightly separated _____. (Not shown that way)

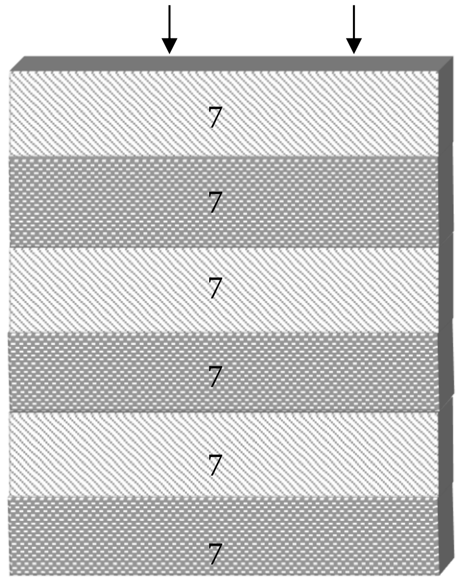


RODS AND STAMPS *Unit 42

4b) If stacked as shown, how many stamp-size square surfaces disappear? _____

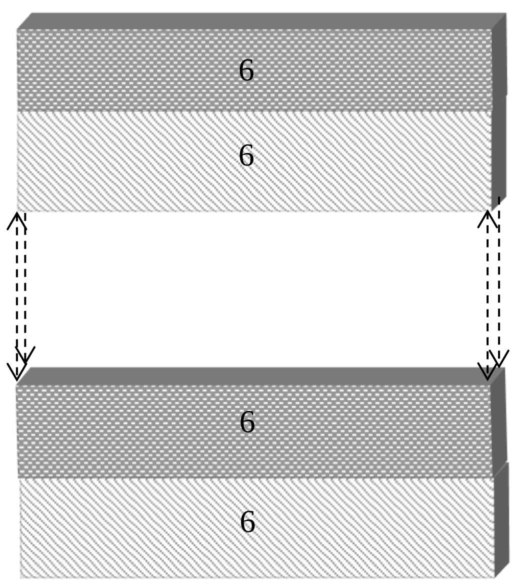
(35, 36, and 70 are likely wrong answers.)

5) How many squares are exposed for these joined logs? ____



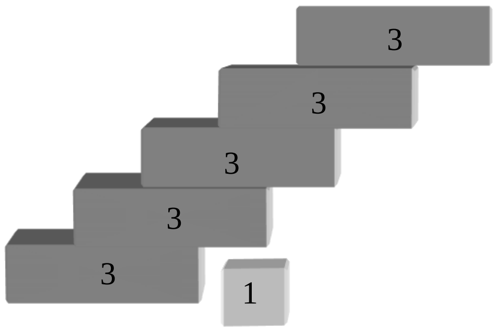
6) How many stamps for a stack of twenty 7-rods? _______

7) How many stamps to cover 1000 6-rods with no separations? _________



RODS AND STAMPS

*Unit 42



8) In the 3-rods above, how many stamps are needed to cover all exposed surfaces including bottoms and backs but not the touching surfaces? (The cube shows that the rods are offset by one cube.) __________

*9) How many stamps would cover 5000 3-rods stacked as above, at right? ______________

10) In the table below, three answers are given to let you know how you are doing as you find the number of stamps needed for each rod length with increasing numbers of rods, no separations.

Don't compute every answer. Use the patterns in the answers to complete the table, but check yourself from time to time by computing.

LENGTH OF EACH ROD

NUMBER OF RODS

	3	4	5	6	7	8	9	10
3								
4			58					
5								
6			82					
7								
8								
9								
10							218	

RODS AND STAMPS *Unit 42

Answers 1 – 10

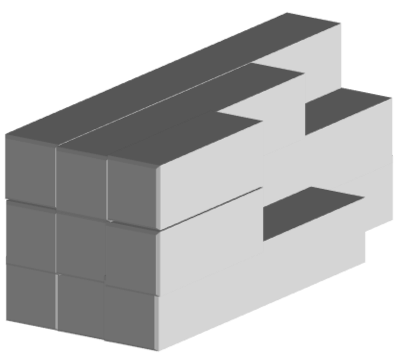
1) 26 **2)** 50 **3a)** 52 **b)** 40

4a) 150 **b)** 56 **5)** 110 **6)** 334 **7)** 14,012 **8)** 54 ***9)** 50,004

10) See below

Length of Each Rod

	3	4	5	6	7	8	9	10
3	30	38	46	54	62	70	78	86
4	38	48	58	68	78	88	98	108
5	46	58	70	82	94	106	118	130
6	54	68	82	96	110	124	138	152
7	62	78	94	110	126	142	158	174
8	70	88	106	124	142	160	178	196
9	78	98	118	138	158	178	198	218
10	86	108	130	152	174	196	218	240

11) 

The stack to the left is lying on its side and the table below it gives the lengths of each rod by telling the number of cubes in its length.

You will see only flat *tables* for stacks in the rest of the exercises.

10	8	4
10	10	4
10	10	8

Tell the total number of *cubes* that were needed to build the stack.

RODS AND STAMPS *Unit 42

12) Number of cubes: _______

9	9	9
9	9	9
9	9	9

In all of the tables below, look for patterns to allow shortcuts instead of simply adding.

8	9	10
8	10	9
9	9	9
9	9	8

13) Note row sums

____

8	8	8	8	8
10	10	10	10	10
8	10	8	10	9
12	6	9	6	12

14) Note rows

____

9	10	12	11
5	6	4	8
7	2	3	1
13	16	15	14

15) Note column sums

____

16) Use the pattern

1	2	3	4	5	6	7	8
16	15	14	13	12	11	10	9

____

17) This is like exercise 15

36	34	32	30	28	26
12	10	8	6	4	2
38	40	42	44	46	48

____

18)

1	3
5	7

____

19)

1	3	5
7	9	11
13	15	17

______

RODS AND STAMPS *Unit 42

20) Note how many odd numbers there are in exercise 18 and 19 above and how that relates to the answer. Use that idea for exercise 20 below.

1	3	5	7	9
11	13	15	17	19
21	23	25	27	29
31	33	35	37	39
41	43	45	47	49

Total number of cubes: _______

Powers of 2**:**

21)

1	1/2
1/4	1/8

____

22) Finish bottom row and give sum.

1	1/2	1/4
1/8	1/16	1/32
_____	_____	_____

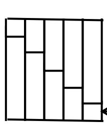
_____

23a) Fill in blanks

1	1/2	2^{-2}	2^{---}	2^{---}	1/32
1/64	1/128				2^{---}

b) The sum of the rod lengths _______

c) How much short of 2 is the sum? _______

24) 

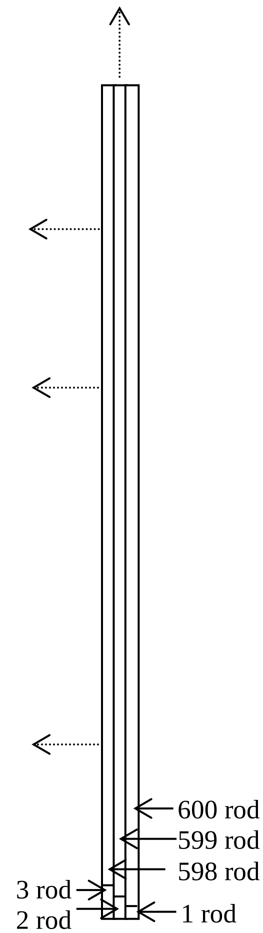
Find the total number of stamps.

1 cube _______

RODS AND STAMPS

*Unit 42

25) This tall stack of rods is like exercise 24. It extends left until the upmost rod is a cube and the lower rod is 600. What is the total number of cubes? _______________



26) Find the sum of these whole numbers:

a) 1 to 16 ______ b) 1 to 1000 _________ c) 1 to 600 ________

In exercises 12 through 14 it was easy to know the average number. In exercise 27 there are twice as many 33's as 24's. So, the average should be twice as close to 33 as to 24. Knowing just that, decide what the average is and use it to find the number of cubes in the stack.

27) The difference between 33 and 24 is 9, so the average is 3 less than 33 and 6 more than 24. The average is _____ and the total number of cubes is _____.

33	33	33	33
33	33	33	33
24	24	24	24

RODS AND STAMPS *Unit 42

28) Number of cubes: _____

28	28	28	28
28	28	28	28
16	16	16	16

29) Number of cubes:_____

28	28	28	28	28
16	16	16	16	16
16	16	16	16	16

30) Number of cubes:_____

42	42	42	42	42	42
30	30	30	30	30	30
42	42	42	42	42	42
42	42	42	42	42	42

Answers 11 – 30

11) 74 **12)** 81 **13)** 107 **14)** 180 **15)** 136

16) 136 (Did you see the connection with ex.15?) **17)** 486 **18)** 16

19) 81(Sum of first nine odds.) **20)** 625 (Sum of first 25 odd numbers.) **21)** 1 7/8

22) 1/64, 1/128, 1/256, 1 255/256 **23a)** $^{-3}$, $^{-4}$, 1/256 or 2^{-8}, 1/512 or 2^{-9}, 1/1024 or 2^{-10} , $^{-11}$.

23b) 1 2047/2048 **b)** 1/2048 or 2^{-11}

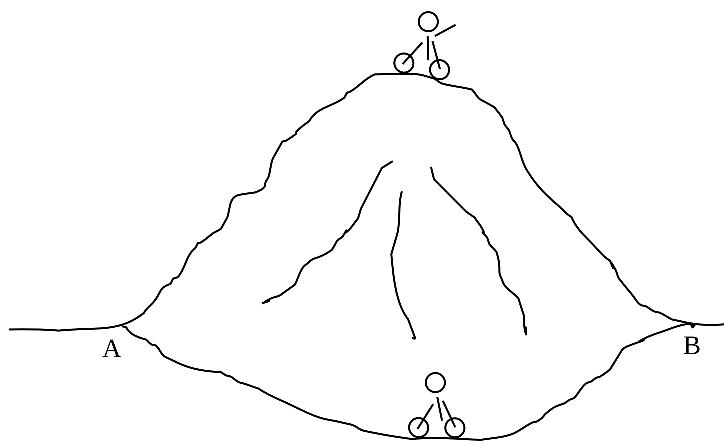
24) 30 **25)** 360,600 (601x600)

26a) 136 (16x17x1/2) **b)** 500,500 (1000/2) x1001 **c)** 180,300 (600/2) x601

27) 30, 360 **28)** 288 **29)** 300 **30)** 936

RODS AND STAMPS *Unit 42

31) Brie and Beck rode their bikes from A to B as shown, two exciting places. Brie's speed up the hill was 5 miles per hour and down the hill was a reckless 45 miles per hour. Beck's speed around the hill (same *distance* as Brie), was at 25 miles per hour. Also, the top of the mountain is the half-way point of Brie's trip. Who reaches exciting place B first, or is it simultaneous? _________ . You might complain that no distance is given. Pick any convenient number if you truly need it.



32) You have had experience above with weighting numbers according to their size and frequency. We see that Beck's speed (rate) is five times that of Brie's climbing rate. While Brie is climbing the hill, Beck goes five times as *far*, or *fully* around the mountain and then on to where she is shown at this moment. **(T/F)** ____
(Of course, from here, Brie will easily beat Beck to interesting point B!)

We know Beck's average speed because it is a constant 25 mph. But finding Brie's average speed for one or more trips is trickier, especially without any distance given. Brie spends 9 times as much *time* climbing as she does descending because her *speeds* are opposite to that (think). We award 9 weights to Brie's climbing time and 1 weight to her descending time for a total of 10 weight units for time: $\big((9\text{x}5)+(1\text{x}45)\big)/10 = 90/10 = 9$ mph average speed for Brie. Beck gets there more quickly but Brie flies high and fast. To each his (her) own.

Answers 31 – 32

31) Beck **32)** T

SQUARE BRACKETS 1 *Unit 43

For all Square Brackets, do Mentally. No Calculator.

> Special thanks to Professor David Page, formerly of the University of Illinois, Urbana, who introduced and demonstrated the use of the greatest integer function to encourage estimating. If there were a Teacher of the Century award, he would get my vote.
>
> PGD

Square Brackets ask for the <u>greatest integer in a number</u>. Study these: $\left\lfloor 6\frac{1}{2} \right\rfloor = 6$. "Square brackets of 6 1/2 equals 6"; or "The greatest integer in 6 1/2 is 6." The greatest integer in 8 3/4 is 8. $\left\lfloor 8\frac{3}{4} \right\rfloor = 8$. The greatest integer in 10 is 10. $[10] = 10$.

Samples: $\left\lfloor 7\frac{3}{4} \right\rfloor = 7$ $\left\lfloor \frac{3}{4} \right\rfloor = 0$ $[14.7] = 14$ $\left\lfloor 1\frac{13}{5} \right\rfloor = \left\lfloor 1 + 2\frac{3}{5} \right\rfloor = 3$

> You are asked for the greatest integer in 1 13/5. Don't just drop the fraction part. Often that works but not always.

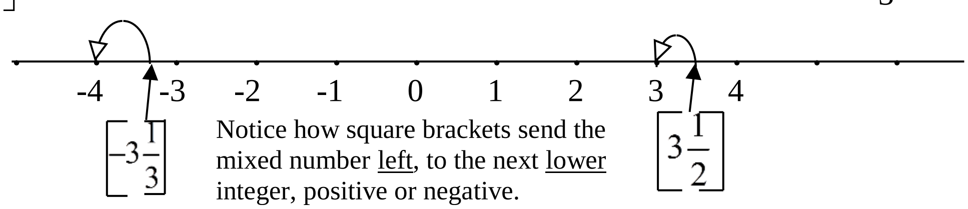
1) $\left\lfloor 6\frac{1}{3} \right\rfloor =$ ___ 2) $\left\lfloor 8\frac{1}{2} \right\rfloor =$ ___ 3) $[5] =$ ____ 4) $\left\lfloor 10\frac{3}{4} \right\rfloor =$ ___

5) $\left\lfloor 96\frac{2}{3} \right\rfloor =$ ___ 6) $\left\lfloor 2\frac{1}{5} \right\rfloor =$ ___ 7) $\left\lfloor \frac{10}{3} \right\rfloor =$ ___ 8) $\left\lfloor 1\frac{13}{4} \right\rfloor =$ ___

9) $\left\lfloor \frac{14}{5} \right\rfloor =$ ___ 10) $\left\lfloor 2\frac{5}{3} \right\rfloor =$ ___ 11) $[1000.04] =$ ________ 12) $\left\lfloor \frac{23}{8} \right\rfloor =$ ___

Pause here and check answers for 1-12 on next page.

13) $\left\lfloor -3\frac{1}{3} \right\rfloor = -4$. Likely wrong answer is ______. The greatest integer in $-3\frac{1}{3}$ is -4.



Also note that $-3\frac{1}{3}$ is less than -3 but greater than -4. $-3\frac{1}{3}$ "contains" -4, but not -3.

14) $\left\lfloor \frac{1}{2} \right\rfloor =$ ____ 15) $\left\lfloor -\frac{1}{2} \right\rfloor =$ ___ 16) $\left\lfloor 1\frac{1}{2} \right\rfloor =$ ___ 17) $\left\lfloor -1\frac{1}{2} \right\rfloor =$ ___ 18) $\left\lfloor 43\frac{2}{3} \right\rfloor =$ ____

19) $\left\lfloor -43\frac{2}{3} \right\rfloor =$ ____ 20) $[-99.8] =$ ____ 21) $\left\lfloor -\frac{9}{11} \right\rfloor =$ ___ 22) $\left\lfloor \frac{5}{2\frac{1}{2}} \right\rfloor =$ ___ 23) $\left\lfloor \frac{5}{2\frac{3}{4}} \right\rfloor =$ ___

SQUARE BRACKETS 1 *Unit 43

24) $\left\lfloor \dfrac{5}{2\frac{1}{3}} \right\rfloor =$ ___ 25) $\left\lfloor \dfrac{1}{1{,}000{,}000} \right\rfloor =$ _____ 26) $\left\lfloor -\dfrac{1}{1{,}000{,}000} \right\rfloor =$ _____

Square brackets give many rich opportunities for estimating by *thinking* about number relationships. Exercise 23 really asks you to see that there are *not quite* two $2\frac{3}{4}$'s in 5. Exercise 24 wants you to see that there are *slightly more* than two $2\frac{1}{3}$'s in 5. The square brackets make it to **your advantage** to estimate by seeing those relationships. There are more than 100 examples in this paper but each one makes a point. Most of your work will be reasoning to avoid computation. The same idea will appear more than once, and often in groups.

Answers 1 – 26

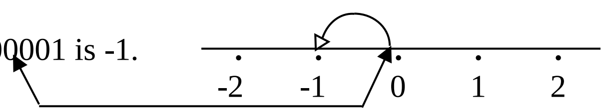
1) 6 **2)** 8 **3)** 5 **4)** 10 **5)** 96 **6)** 2 **7)** $\left\lfloor \dfrac{10}{3} \right\rfloor = \left\lfloor 3\frac{1}{3} \right\rfloor = 3$ **8)** 4

9) 2 **10)** $\left\lfloor 2\dfrac{5}{3} \right\rfloor = \left\lfloor \dfrac{11}{3} \right\rfloor = \left\lfloor 3\dfrac{2}{3} \right\rfloor = 3$ **11)** 1000 **12)** 2

13) -3 is the likely wrong answer. (The true answer is -4.) **14)** 0 **15)** -1 **16)** 1 **17)** -2

18) 43 **19)** -44 **20)** -100 (not -99) **21)** -1 **22)** 2 **23)** 1 **24)** 2 **25)** 0 **26)** -1

The square brackets of a number a tiny bit below zero is -1. The greatest integer in **-.**0000000000000000000001 is -1.



27) $\left\lfloor -\dfrac{1}{10000000000000000000000} \right\rfloor =$ ________ 28) $[-.01] =$ _____

<u>Reminder</u>: $.\overline{9}$ **=** .9999 . . . = 1 ← We saw this in Achilles.

Shortcut reminder for repeating decimals:
Just as $.\overline{29} = 29/99$, so does $.\overline{9} = 9/9 = 1$,
(Unit 30) and $.\overline{99}\ldots = 99/99 = 1$

29) $\left\lfloor 8\dfrac{17}{8} \right\rfloor =$ ____ 30) $[.9] =$ ___ 31) $[.999999] =$ ___ 32) $[.99\ .\ .\ .] =$ ___

33) $[.\overline{9}] =$ ___ 34) $[28.9999\ .\ .\ .] =$ ____ 35) $[-.\overline{9}] =$ ____ 36) $[10^{-400}] =$ ___

37) $\left\lfloor 8\dfrac{7}{3} \right\rfloor =$ ___ 38) $[6.999] =$ ___ 39) $[6.999\ .\ .\ .] =$ ___ 40) $\left\lfloor \dfrac{14.23}{4.1} \right\rfloor =$ ___

Pause here and check answers for 29 - 40

SQUARE BRACKETS 1 *Unit 43

1/2 + a number near ½, is going to give an answer near 1. The exact amount is not needed .This sum barely exceeds 1 because 7/13 is slightly greater than 1/2.

<u>Samples</u>: $\left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{7}{13} \right\rfloor = 0 + 0 = 0$ $\quad \left\lfloor \frac{1}{2} + \frac{7}{13} \right\rfloor = 1$

41) $\left\lfloor \frac{19}{20} + \frac{1}{21} \right\rfloor =$ ___ 42) $\left\lfloor 1\frac{6}{5} \right\rfloor =$ ___ 43) $\left\lfloor \frac{14}{15} + \frac{1}{17} \right\rfloor =$ ___ 44) $\left\lfloor \frac{1}{2} + \frac{7}{15} \right\rfloor =$ ___

45) $\left\lfloor \frac{27\frac{9}{10}}{7} \right\rfloor =$ ___ 46) $\left[\sqrt{35} \right] =$ ___ 47) $\left\lfloor \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \right\rfloor =$ ___ 48) $\left\lfloor 6 - 3\frac{3}{7} \right\rfloor =$ ___

49) $\left\lfloor -\frac{9}{11} \right\rfloor =$ ___ 50) $\left\lfloor -\frac{11}{9} \right\rfloor =$ ___ 51) $\left\lfloor \frac{172}{90} \right\rfloor =$ ___ 52) $\left\lfloor -\frac{172}{90} \right\rfloor =$ ___

$\left[10^{-400} \right] = \left\lfloor \frac{1}{10^{400}} \right\rfloor =$

Answers 27 – 52

27) -1 **28)** -1 **29)**10 **30)** 0 **31)** 0

32) 1 **33)** 1 **34)** 29 **35)** -1 **36)** 0

37) 10 **38)** 6 **39)** 7 **40)** 3

---

41) 0 **42)** 2 **43)** 0 **44)** 0 **45)** 3 **46)** 5

47) 0 **48)** 2 **49)** -1 **50)** -2 **51)** 1 **52)** -2, not -1

Note on 47: If fractions had been 1/4+1/4+1/4+1/4, sum would =1.

If you made errors with negatives, visualize the problem on the number line.

SQUARE BRACKETS 2 **Unit 44

No calculator   If a question is **not fair** without a calculator, answer NF.
Otherwise give correct answer.

1/5 of 4 is less than 1, but 1/4 of 5 is not.

Many of these are not easy. Take your time.

Thinking is top priority.

$\left\lfloor \frac{1}{17}\times 18 \right\rfloor = 1$ $\left\lfloor \frac{1}{18}\times 17 \right\rfloor = 0$ $\left\lfloor 5\frac{1}{5}\times 4 \right\rfloor = 20$ $\left\lfloor 4\frac{1}{4}\times 5 \right\rfloor = 21$

Compare approximately to 1/20 x 40

1) $\left\lfloor \frac{1}{20}\times 41 \right\rfloor =$ ___ 2) $\left\lfloor \frac{1}{21}\times 40 \right\rfloor =$ ___ 3) $\left\lfloor \frac{3}{25}\times 76 \right\rfloor =$ ___ 4) $\left\lfloor \frac{3}{25}\times 74 \right\rfloor =$ ___

5) $\left\lfloor \frac{19}{20}+\frac{1}{17} \right\rfloor =$ ___ 6) $\left\lfloor 2\frac{1}{16}\times 17 \right\rfloor =$ ___ 7) $\left\lfloor 2\frac{1}{18}\times 17 \right\rfloor =$ ___ 8) $\left\lfloor 2\frac{1}{89}\times 90 \right\rfloor =$ ___

One fraction is greater than 1, the other is less than 1, but not equally so.

9) $\left\lfloor \frac{14}{15}+\frac{15}{14} \right\rfloor =$ ___ 10) $\left\lfloor \frac{21}{20}+\frac{20}{21} \right\rfloor =$ ___ 11) $\left\lfloor \frac{11}{16}+\frac{13}{15} \right\rfloor =$ ___ 12) $\left\lfloor \frac{11}{16} \right\rfloor + \left\lfloor \frac{13}{15} \right\rfloor =$ ___

13/15 is 2/15 less than1, but . .

13) $\left\lfloor \frac{13}{15}+\frac{16}{14} \right\rfloor =$ ___ *14 $\left\lfloor \frac{1}{1000}+1.9899 \right\rfloor =$ ___ 15) $\left\lfloor \frac{1}{30}+\frac{29}{31} \right\rfloor =$ ___

16) $\left\lfloor \frac{1}{30}+\frac{30}{31} \right\rfloor =$ ___ 17) $\left\lfloor (.98)^2 \right\rfloor =$ ___ 18) $\left\lfloor (.99)^{2000} \right\rfloor =$ ___ 19) $\left[(1.02)^3 \right]^{500} =$ ___

First, Estimate $(1.02)^3$.

20) $\left\lfloor \frac{7}{34}+\frac{7}{35}+\frac{7}{36}+\frac{7}{37}+\frac{7}{38} \right\rfloor =$ ___ *21) $\left[(1.02)^{500} \right]^3 =$ ___ **22) $\left\lfloor \frac{6}{17}-\frac{7}{18} \right\rfloor =$ ___

Hint: Which fraction is closer to 1??

SQUARE BRACKETS 2 **Unit 44

Correct to here now; answers on page 4 of this unit.

23) $\left\lfloor \dfrac{4.9}{2\frac{1}{2}} \right\rfloor =$ ___ 24) $\left[\left(1\dfrac{1}{10}\right)\right]^{100} =$ ____ 25) $\left[\left(\dfrac{1}{10}\right)\right]^{100} =$ ____ 26) $\left[\left(1\dfrac{1}{10}\right)^{100}\right] =$ ___

*27) $\left[16.\overline{9} - 16.9\right] =$ ___ *28) $\left[16.9 - 16.\overline{9}\right] =$ ____ 29) $\left\lfloor \dfrac{300.44}{600.9} + 5 \right\rfloor =$ ____

30) $\left\lfloor \dfrac{1}{275} - \dfrac{1}{274} \right\rfloor =$ ___ 31) $\left\lfloor 2\dfrac{1}{89} \times 90 \right\rfloor =$ _____ 32) $\left\lfloor \dfrac{41}{275} \div \dfrac{41}{274} \right\rfloor =$ _____

33) Is $\left[(1.02)^{100}\right]$ = ? a fair question without a calculator? ____ 34) Is exercise 33 fair? ____

Correct to here now; answers on page 4 of this unit.

In $\left\lfloor \dfrac{4}{9} + \dfrac{7}{13} \right\rfloor = ?$, each fraction is close to 1/2. 4/9 falls short of 1/2, 7/13 exceeds 1/2. We need to know which "wins". Each of the following is exactly 1/2: $\left\lfloor \dfrac{4\frac{1}{2}}{9} + \dfrac{6\frac{1}{2}}{13} \right\rfloor$**.** Compare to $\left\lfloor \dfrac{4}{9} + \dfrac{7}{13} \right\rfloor$. 4/9 is 1/2 <u>ninth</u> <u>less</u> than 1/2; 7/13 is only 1/2 <u>thirteenth</u> <u>greater</u> than 1/2**.** 4/9 "wins"**:** the lack is greater than the excess**.** $\left\lfloor \dfrac{4}{9} + \dfrac{7}{13} \right\rfloor = 0.$

*35) $\left\lfloor \dfrac{21}{40} + \dfrac{20}{41} \right\rfloor =$ ___ *36) $\left\lfloor \dfrac{62}{125} + \dfrac{58}{114} \right\rfloor =$ ___ 37) $\left\lfloor \dfrac{1}{179} \div \dfrac{1}{180} \right\rfloor =$ ___

Be watchful for patterns or relationships. For instance, 3/4 times something is less than 4/5 of it, which is less than 99/100 which is more than 98/99**.** But don't forget that 1/4 of something is greater than 1/5 of it.

38) $\left\lfloor \dfrac{98}{99} \times \dfrac{99}{98} \right\rfloor =$ ___ * 39) $\left\lfloor 1\dfrac{1}{97} \times \dfrac{98}{99} \right\rfloor =$ ___ 40) $\left\lfloor \dfrac{98}{99} \times \dfrac{87}{88} \right\rfloor =$ ___

SQUARE BRACKETS 2 **Unit 44

Recall that 20 x 20 = ____,
but 19 x 21 = ____. Use in #41

*41) $\left\lfloor \frac{98}{99} \times \frac{98}{97} \right\rfloor = $ ___ 42) $\left\lfloor \frac{198}{199} \times \frac{189}{198} \right\rfloor = $ ___ 43) $\left\lfloor \frac{98}{99} \times \frac{101}{100} \right\rfloor = $ ___

Multiply tops and bottoms mentally.
Hint:
17 x 101 = 1717

Note: $\sqrt{1.21}$ *is closely related to* $\sqrt{121}$ *but* $\sqrt{12.1}$ *is not.*

44) $\left\lfloor \sqrt{138} \right\rfloor = $ ____ 45) $\left\lfloor \sqrt{121} \right\rfloor = $ ___ *46) $\left\lfloor \sqrt{1.21} \right\rfloor = $ ___ 47) $\left\lfloor \sqrt{12.1} \right\rfloor = $ ___

48) $\left\lfloor \sqrt{.121} \right\rfloor = $ ____ 49) $\left\lfloor \sqrt{9.1} \right\rfloor = $ ___ * 50) $\left\lfloor \sqrt{\left[\sqrt{9.1}\right]} \right\rfloor = $ ___

Work from the Inside out

51) $\left\lfloor \sqrt{.90} \right\rfloor = $ ____ 52) Use calculator to find: $\sqrt{\sqrt{.99}}$ _______

53) $\left\lfloor \sqrt{\sqrt{\sqrt{.98}}} \right\rfloor = $ ____ 54) $\left\lfloor \sqrt{\sqrt{\sqrt{4}}} \right\rfloor = $ ____ 55) $\left[\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{.99}}}}}}}} \right] = $ ____

Exercise 55 might seem unfair but it isn't. Take a guess. Then check with calculator.

*56) $\left\lfloor \frac{49}{100} + \frac{50}{99} \right\rfloor = $ ____ 57) $\left[\frac{44.\overline{9}}{9.\overline{9}} \right] = $ ____ 58) $\left\lfloor 18\frac{20}{21} \div 18\frac{19}{20} \right\rfloor = $ ____

59) $\left\lfloor \frac{1681^4 \times \frac{3}{4}}{1681^4 \times \frac{4}{5}} \right\rfloor = $ ____ *60) $\left\lfloor 8\frac{22}{23} - 8\frac{23}{24} \right\rfloor = $ ____ 61) $\left\lfloor \frac{\left\lfloor \sqrt{79} \right\rfloor}{8\frac{1}{3}} \right\rfloor = $ ____

First perform the two indicated operations on the numerator.

62) $\left\lfloor 6\frac{12}{13} - 2\frac{18}{19} \right\rfloor = $ ____ *63) $\left\lfloor \frac{\sqrt{63}}{7.3} \right\rfloor = $ ____ 64) $\left\lfloor \frac{\left\lfloor \sqrt{63} \right\rfloor}{7.3} \right\rfloor = $ _____

65) $\left\lfloor \frac{7}{8} + \frac{1}{7} \right\rfloor + \frac{5}{6} = $ ______ 66) $\frac{7}{8} + \left\lfloor \frac{1}{7} + \frac{5}{6} \right\rfloor = $ ______ * 67) $\left\lfloor \frac{15}{17} - \frac{17}{19} \right\rfloor = $ _____

Hint: Ask how far each fraction falls short of 1.

68) $\left\lfloor \frac{13}{14} - \frac{17}{19} \right\rfloor = $ _____ * 69) $\left\lfloor \frac{40}{49} - \frac{23}{32} \right\rfloor = $ _____ 70) $\left\lfloor \frac{7}{8} - \frac{97}{99} \right\rfloor = $ _____

SQUARE BRACKETS 2 **Unit 44

71) You probably know that in ordinary arithmetic, adding three numbers like (1 + 2) + 3 and 1 + (2 + 3) gets you the same answer. So addition is said to be *associative* because you get the same answer whether you *associate* the first two numbers or the last two numbers. What have examples 65 and 66 told you?

____

____

72) $\left[\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{1.01}}}}}}}}\right] =$ ____ Okay to check with calculator.

Answers

1) 2 **2)** 1 **3)** 9 **4)** 8 **5)** 1 **6)** 35 **7)** 34 **8)** 181

9) 2 **10)** 2 **11)** 1 **12)** 0 **13)** 2 **14)** 1 **15)** 0 **16)** 1

17) 0 **18)** 0 **19)** 1 **20)** 0

21) NF. You cannot find the value of $(1.02)^{500}$ w/o a calculator. ***22)** -1

Return to text.

23) 1 **24)** 1 **25)** 0 **26)** NF. $(1\ 1/10)^{100}$ needs calculator. **27)** 0

28) -1 **29)** 5 **30)** -1 **31)** 181 **32)** 0 **33)** No **34)** Yes

Return to text

***35)** 1 ***36)** 1 **37)** 1 **38)** 1 ***39)** 1 **40)** 0

Return to text.

***41)** 1 **42)** 0 ***43)** 0 **44)** 11 **45)** 11 ***46)** 1 **47)** 3 **48)** 0

49) 3 ***50)** 1 **51)** 0 **52)** .9975 **53)** 0 **54)** 1 **55)** 0 ***56)** 0

57) 4 **58)** 1 **59)** 0 ***60)** -1 **61)** 0 **62)** 3 ***63)** 1 **64)** 0

65) 1 5/6 **66)** 7/8 **67)** -1 **68)** 0 **69)** 0 **70)** -1

71) Exercises 65 and 66 told you that addition with square brackets is not <u>always</u> associative, only sometimes. **72)** 1

Simultaneous Equations *Unit 45

In the same problem, the same shape must have the same number. All of the “sames” that appear below account for the term “simultaneous”.

1) $\begin{cases} \bigcirc \times \triangle = 20 \\ \bigcirc + \triangle = 9 \end{cases}$ 2) $\begin{cases} \bigcirc \times \triangle = 20 \\ \bigcirc + \triangle = 12 \end{cases}$ 3) $\begin{cases} \bigcirc \times \triangle = 20 \\ \bigcirc + \triangle = 21 \end{cases}$

In the same problem, different shapes may have the same or different numbers.

4) $\begin{cases} \bigcirc + \triangle = 18 \\ \bigcirc \times \triangle = 81 \end{cases}$ 5) $\begin{cases} \bigcirc \times \triangle = 81 \\ \bigcirc + \triangle = 30 \end{cases}$ 6) $\begin{cases} \bigcirc \times \triangle = 81 \\ \bigcirc + \triangle = 82 \end{cases}$

You might see a strategy. Try multiplying *first*, *then* find an add or subtract to agree.

7) $\begin{cases} \bigcirc + \triangle = 12 \\ \bigcirc \times \triangle = 20 \end{cases}$ 8) $\begin{cases} \bigcirc - \triangle = 1 \\ \bigcirc \times \triangle = 20 \end{cases}$ 9) $\begin{cases} \bigcirc \times \triangle = 20 \\ \bigcirc + \triangle = 21 \end{cases}$

When the factors are close or equal, the products are maximal, or nearly so.

*10) $\begin{cases} \bigcirc \times \triangle = 72 \\ \bigcirc \div \triangle = 8 \end{cases}$ 11) $\begin{cases} \bigcirc \times \triangle = 120 \\ \bigcirc - \triangle = 2 \end{cases}$ *12) $\begin{cases} \bigcirc \times \triangle = 143 \\ \bigcirc - \triangle = 2 \end{cases}$

13) $\begin{cases} \bigcirc + \triangle = 15 \\ \bigcirc \times \triangle = 54 \end{cases}$ 14) $\begin{cases} \bigcirc \times \triangle = 400 \\ \bigcirc \div \triangle = 1 \end{cases}$ *15 $\begin{cases} \square \times \text{⯃} = 32 \\ \square \div \text{⯃} = 8 \end{cases}$

*16 $\begin{cases} \text{⬡} + \triangle + \triangle = 19 \\ \text{⬡} \times \triangle = 39 \end{cases}$ *17) $\begin{cases} \bigcirc \times \triangle = 289 \\ [\bigcirc - 1] \times [\bigcirc + 1] = 288 \end{cases}$ *18 $\begin{cases} \bigcirc \times \triangle = 110 \\ \bigcirc^2 + \triangle^2 = 221 \end{cases}$

Note: All triangles have same number. Same number in all circles.

SIMULTANEOUS EQUATIONS *Unit 45

19) $\begin{cases} \bigcirc^2 + \triangle^2 = 25 \\ \bigcirc \text{ x } \triangle = 12 \end{cases}$ **20)** $\begin{cases} \bigcirc^2 - \triangle^2 = 13 \\ \bigcirc \text{ x } \triangle = 42 \end{cases}$ ***21)** $\begin{cases} \square + \text{⯃} = 13 \\ \square - \text{⯃} = 5 \end{cases}$

When we expand to include the positive and negative rational numbers, we must look at a greater variety of possibilities: Exercise 23 has more numbers that "work" than exercise 19. Same for 20 and 24.

*22) $\begin{cases} \bigcirc + \triangle = 4\tfrac{1}{2} \\ \bigcirc \text{ x } \triangle = 2 \end{cases}$ **23)** $\begin{cases} \bigcirc^2 + \triangle^2 = 25 \\ \bigcirc \text{ x } \triangle = 12 \end{cases}$ ***24)** $\begin{cases} \square^2 - \text{⯃}^2 = 95 \\ \square \text{ x } \text{⯃} = 84 \end{cases}$

25) $\begin{cases} \bigcirc + \triangle = 0 \\ \bigcirc \text{ x } \triangle = 0 \end{cases}$ **26)** $\begin{cases} \bigcirc + \triangle = 0 \\ \bigcirc \text{ x } \triangle = -25 \end{cases}$ 27) $\begin{cases} \bigcirc + \text{⯃} = 0 \\ \text{⯃}^2 = 36 \end{cases}$

***28)** $\begin{cases} \bigcirc + 2\text{x} \text{⬠} = 11 \\ \bigcirc \text{ x } \text{⬠} = 14 \end{cases}$

29) Stare at this until a solution occurs to you: $\sqrt{2} \text{ x } \sqrt{2} =$ ___.

30) Now try this: $2 \div \sqrt{2} =$ _____

The computer's calculator tells us that $\sqrt{2}$ = 1.4142135623730950488016887242097. . .. This is not *exactly* true, of course, because if we multiplied the long decimal by itself (squared it), we would get a decimal about twice as long and very close to 2, but it would end in 9. You will no doubt agree, even though the computer's calculator would give 2 as the result if you used it to square the above approximation.

The *fact* that can boggle the mind is that this infinitely long decimal (not the approximation), multiplied by itself, does equal exactly 2. Expanding from the *rational* numbers to the *algebraic* numbers (see *The Abyss)*, we have:

***31)** $\begin{cases} \square \div \bigcirc = \sqrt{3} \\ \bigcirc^2 = 3 \end{cases}$

SIMULTANEOUS EQUATIONS *Unit 45

Answers

The answers are given as pairs in either order if it does not matter which number is assigned to a particular box shape. If the order *does* matter, the *single* pair given is the only correct answer.

1) 4, 5 or 5, 4	**2)** 2, 10 or 10, 2	**3)** 20, 1 or 1, 20
4) 9, 9	**5)** 3, 27 or 27, 3	**6)** 1, 81 or 81, 1
7) 2, 10 or 10, 2	**8)** 5, 4 (Not 4, 5)	**9)** 1, 20 or 20, 1
10) 24, 3	**11)** 12, 10	**12)** 13, 11
13) 9, 6 or 6, 9	**14)** 20, 20	**15)** 16, 2
16) 13, 3	**17)** 17, 17	**18)** 10, 11 or 11, 10
19) 3, 4 or 4, 3	**20)** 7, 6	**21)** 9, 4
22) 4, ½ or ½, 4	**23)** 3, 4 or 4, 3 or -3, -4 or -4, -3	**24)** 12, 7
25) 0, 0	**26)** 5, -5 or -5, 5	**27)** 6, -6 or -6, 6
28) 7, 2 or 4, 3 ½	**29)** 2	**30)** $\sqrt{2}$
31) 3, $\sqrt{3}$		

SQUARE BRACKETS 3, ON THE NUMBER LINE **Unit 46

Circle each number in the brackets which, when put in the box, makes the statement true. Study the sample thoughtfully.

Sample: $[\square] = 5$ $\{$ 5, 6, 5.5, 4.999 . . ., 6.003, 4.9999, $3\frac{7}{3}$ $\}$

1) $[\square] = 8$ $\{$ $8\frac{7}{8}$, $8\frac{9}{8}$, 850%, $8.\overline{9}$, 7.5, $7.\overline{9}$ $\}$

2) a. What is the smallest number which would work in exercise 1? ___
(It might not be among the choices.)

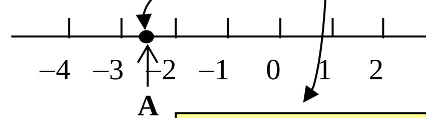
b. What is the largest number which would work in exercise 1 if it exists? _______

3) $[\square + 7] = 12$ $\{$ 4.9, $4.\overline{9}$, 4.5, 5.4, $5.\overline{9}$, 5.0001, 5.999 $\}$

Very important

4) $[\square + 7] = 12.5$ $\{$ 4.9, $4.\overline{9}$, 4.5, 5.4, $5.\overline{9}$, 5.0001, 5.999 $\}$

5) $[\square] = -3$ $\{$ -2.9, -3, –3.1, 3.1, $-2.\overline{9}$, –2.1, -2 $\}$



Correct exercises 1 – 5 now.

The greatest integer in the vicinity of -2.6, such as:
–2, is *greater* than –3.
–2, is too big to be in –2.6.
–3, is the *greatest* integer in -2.6.

6) $6 \times [\square] = 24$ $\{$ 4, 5, 6, $4\frac{1}{2}$, $5\frac{1}{2}$, $6\frac{1}{2}$, 4.2, 4.9, 4.01, 4 1/8 $\}$

7) $[\square] \times 8 = 16$ $\{$ 2, 3, 1, 2 1/8, 2.1, 2 1/9, 2 1/7 $\}$

8) $[\square \times 8] = 16$ $\{$ 2, 3, 1, 2 1/8, 2.1, 2 1/9, 2 1/7 $\}$

9) $5 + [\square] = -3$ $\{$ -2, 2, 2 1/2, 6, -6, -8, -8 1/2, -7 1/2, -7, -2 1/2, -7.2 $\}$

10) $[\square \div 3] = 6$ $\{$ 18, 16, 17 1/2, 18 1/2, 20, 21, 21 1/2, 20.99, 6 $\}$

SQUARE BRACKETS 3, ON THE NUMBER LINE **Unit 46

When more than one box of the same shape is in the same problem, the *same number* must be put in each box. $\square + \square = 10$ is allowed only for 5 and 5, not 6 and 4, nor 9 and 1, etc.

11) $\left[\square + \square\right] = 8$ $\{$ 4.3, 4.6, 4.9, 3.9, 3.99 $\}$

12) $\left[2 \text{ x } \square\right] = 5$ $\{$ 1, 2, 3, 2.4, $\sqrt{8}$, 2.6, $\sqrt{9}$, $2.499\overline{9}$, $2.\overline{9}$, $2.88888\overline{9}$ $\}$

13) $\left[\square + 1/3\right] = \square + 1/3$ $\{$ 10, 1/3, 8 2/3, -1/3, -2/3, -7 2/3, -6 1/3 $\}$

Correct Exercises 6 – 13 now.

Answers 1 – 5

NO!

YES!

1) $8\frac{7}{8}$, $8\frac{9}{8}$, 850%, $8.\overline{9}$, 7.5, $7.\overline{9}$

2a) What is the smallest number which would work in ex.1? **8**
It might not be among the choices.

b) What is the largest number which would work in ex. 1 if it exists? **No such number**.
"The last number before 9" cannot be specified successfully. No matter how close you get, you can always find the average of that number and 9. It is between them.

3) $\left[\square + 7\right] = 12$ $\{$ 4.9, $4.\overline{9}$, 4.5, 5.4, $5.\overline{9}$, 5.0001, 5.999 $\}$

4) $\left[\square + 7\right] = 12.5$ None. Only integers emerge from square brackets.

***5)** $\left[\square\right] = -3$ $\{$ -2.9, -3, -3.1, 3.1, $-2.\overline{9}$, -2.1, -2 $\}$

Recall that –2.9 is *larger* than –3.

Square Brackets 3, On the Number Line **Unit 46

Answers 6 – 13

6) $6 \times [\,\square\,] = 24$ $\{4, 5, 6, 4\frac{1}{2}, 5\frac{1}{2}, 6\frac{1}{2}, 4.2, 4.9, 4.01, 4\ 1/8\}$

7) $[\,\square\,] \times 8 = 16$ $\{2, 3, 1, 2\ 1/8, 2.1, 2\ 1/9, 2\ 1/7\}$

8) $[\,\square \times 8\,] = 16$ $\{2, 3, 1, 2\ 1/8, 2.1, 2\ 1/9, 2\ 1/7\}$ (2 1/7 is too large.)

9) $5 + [\,\square\,] = -3$ $\{-2, 2, 2\ 1/2, 6, -6, -8, -8\ 1/2, -7\ 1/2, 7, -2\ 1/2, -7.2\}$

10) $[\,\square \div 3\,] = 6$ $\{18, 16, 17\ 1/2, 18\ 1/2, 20, 21, 21\ 1/2, 20.99, 6\}$

When more than one box of the same shape is in the same problem, the same number must be put in each box. $\square + \square = 10$ will work only for 5 and 5, not 6 and 4.

11) $[\,\square + \square\,] = 8$ $\{4.3, 4.6, 4.9, 3.9, 3.99\}$

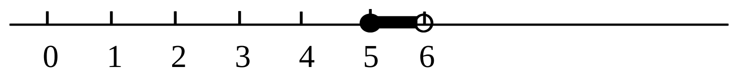
*12) $[\,2 \times \square\,] = 5$ $\{1, 2, 3, 2.4, \sqrt{8}, 2.6, \sqrt{9}, 2.499\overline{9}, 2.\overline{9}, 2.88888\overline{9}\}$

*13) $[\,\square + 1/3\,] = \square + 1/3$ $\{10, 1/3, 8\ 2/3, -1/3, -2/3, -7\ 2/3, -6\ 1/3\}$

SQUARE BRACKETS 3, ON THE NUMBER LINE **Unit 46

Using the number line allows the expressing of *all* the real numbers that make a particular equation work.

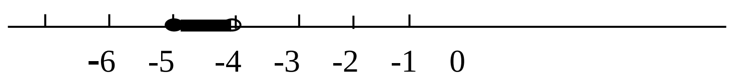
Sample 1: $[\square] = 5$



The symbols have common sense meanings. The graph shows *all real* numbers from 5 (including 5) to 6 (not including 6). The space around 6 is for emphasis, and to show that 6 is not included. An infinite number of points close to 6 do not show, but do work, such as 5.99999999999999999 and 5.98.

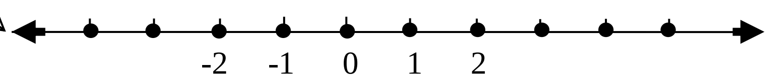
Try to study this until it seems O.K.

Sample 2: $[\square]$ **= -** 5



Sample 3: $[\square] = \square$

note



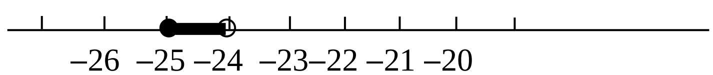
note

The heavy dots ("points") and black arrows show that this equation works for all integers.

Sample 4: $\left[\dfrac{\square}{7}\right] = 3$

Sample 3 shows that all real numbers from 21 to 28 work, including 21 but *not* including 28.Verify all this mentally.

Sample 5: $[18 + \square] = -4$

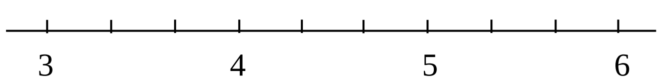


For each of the following, draw graphs similar to the samples, thus showing *all* the real numbers that work for each equation. Use trial and error by trying out likely numbers, but let your successes and misses <u>inform</u> you about the whole set of numbers that work. You are given a number line without numbers. Insert numbers according to need.

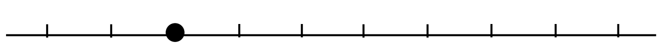
14) $3 \times [\square] = 12$

Hints: 4 5

*15) $[3 \times \square] = 12$



*16) $[5 + \square] = -3$



Hint: –8

SQUARE BRACKETS 3, ON THE NUMBER LINE **Unit 46

*17) $6 - \left[\square\right] = 4$

*18) $6\frac{1}{2} - \left[\square\right] = 4$

Insert numbers below each line according to need

19) $\left[\square\right] = -10$

*20) $4 + \left[\square\right] = -3$

**21) $\left[4 + \square\right] = -3$

*22) $\left[\dfrac{\square}{4}\right] = 6$

*23) $\left[\dfrac{\square + 3}{4}\right] = 6\frac{1}{2}$

*24) $\left[\square\right] + \frac{1}{2} = \square + \frac{1}{2}$

**25) $\square + \frac{1}{2} = \left[\square + \frac{1}{2}\right]$

SQUARE BRACKETS 3, ON THE NUMBER LINE **Unit 46

*26) $\left[\dfrac{\square}{7}\right] = 8$

*27) $\left[\dfrac{\square}{7}\right] = -8$

28) $\left[\dfrac{\square + \square}{2}\right] = \left[\square\right]$

***29) These are not easy. Be sure to try negatives. Graph on the real number line. The strategy is to try numbers that are likely to work, but try zero first, usually.

> Notice as you try numbers to see if they work, that the denominators are closely related to the endpoints of the segment "pieces" that show with successful number tries.

$\left[\dfrac{\square}{3}\right] = \left[\dfrac{\square}{4}\right]$

Reminder: $\left\lfloor -2\dfrac{1}{3} \right\rfloor = -3$

0

Once you have fully corrected exercise 29, you can find some key patterns which will suggest similar (but probably not *equal*) patterns for exercise 30.

***30) $\left[\dfrac{\square}{4}\right] = \left[\dfrac{\square}{5}\right]$

SQUARE BRACKETS 3, ON THE NUMBER LINE

**Unit 46

Answers 14 – 30

14) $3 \times \left[\square\right] = 12$

Hints: 4 5

***15)** $\left[3 \times \square\right] = 12$

3 4 4 1/3 5 6

Note: 3 x 4 1/3 = 13

****16)** $\left[5 + \square\right] = -3$

Hint: –8 –7

Remember, -8 is larger than -8.6

***17)** $6 - \left[\square\right] = 4$

2 3

***18)** $6\frac{1}{2} - \left[\square\right] = 4$

No number works

***19)** $\left[\square\right] = -10$

–10 –9

***20)** $4 + \left[\square\right] = -3$

–7 –6

***21)** $\left[4 + \square\right] = -3$

–7 –6

***22)** $\left[\dfrac{\square}{4}\right] = 6$

24 28

23) $\left[\dfrac{\square + 3}{4}\right] = 6\frac{1}{2}$

No number works for this expression.

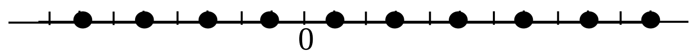
24) $\left[\square\right] + \frac{1}{2} = \square + \frac{1}{2}$

All integers work for this expression.

SQUARE BRACKETS 3, ON THE NUMBER LINE

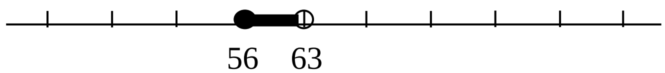
****Unit 46**

25) $\Box + \frac{1}{2} = \left[\Box + \frac{1}{2}\right]$

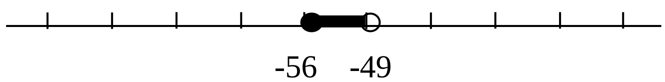


All (integers-plus-one-half) numbers make ex. 25 work.

26) $\left[\frac{\Box}{7}\right] = 8$



27) $\left[\frac{\Box}{7}\right] = -8$



28) $\left[\frac{\Box + \Box}{2}\right] = [\Box]$

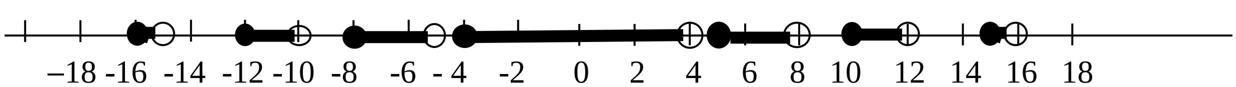
All real Numbers

*****29)**

$\text{——} = \left[\frac{\Box}{4}\right]$

*****30)**

$\left[\frac{\Box}{4}\right] = \left[\frac{\Box}{5}\right]$



RAMANUJAN **Unit 47

<u>Prerequisites</u>: Units 19 and 39, Halfwalk and To End or Not to End?

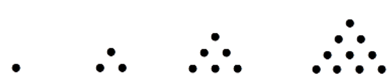
Srinawasa Ramanujan (Shree na vasa Rah mah nu jan) was an extraordinary mathematical talent born in India in 1877. Although he received many awards for mathematics achievement in school he was so absorbed in that favorite activity that he did not do well in his other subjects. Even so, it was many years before his skills would be properly recognized. He received practically no formal advanced training in mathematics until he caught the attention of the famous English mathematician Godfrey Hardy.

Even before that time, however, Ramanujan did significant original work which he kept in a series of notebooks. As discoveries of his work by other mathematicians unfolded, there was much wonder and admiration that an untrained person could produce such remarkable work. Some even rated his abilities as comparable to those of Gauss, Euler, Newton and Archimedes. Excellent accounts of his life and work are given in a book we so often seem to turn to, Calvin Clawson's *Mathematical Mysteries*, #6 on our book list, and in the Wikipedia article accessible merely by searching <Ramanujan>.

Although much in the articles mentioned is advanced work, you can get a good sense of the kinds of things Ramanujan gave to the world by skirting around those discussions which mean little to most non-mathematicians.

The mathematics that follows is constructed from some scribbled notes made by the author many years ago. I regret not being able to give sources.

Here is a well-known sequence of numbers called the triangular numbers:

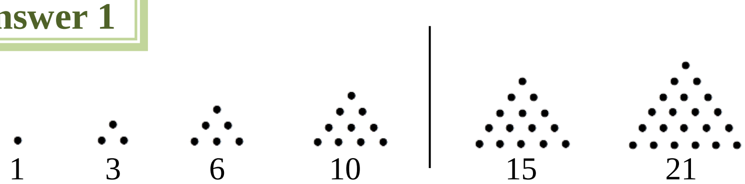


1   3   6   10   ____   ____   ____   ____   ____

1) Extend the above sequence for two more diagrams and five more numbers.

Answer 1

1)



1 3 6 10 15 21 28 36 45

2) Here are two series you know about :

a) $\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$ **b)** $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$

The first is in the *Halfwalk* unit and the second is in the unit *To End or not to End?*

*a) What is the 100th term of the first series. (Be careful of a very likely wrong answer to this exercise). Use an exponent in your answer.________

b) What is the 100th term of the second series (the *harmonic* series) ________

c) With some thought, you can determine (or remember from unit 19) that the Halfwalk series above is convergent. The Harmonic series above was shown in Unit 39 to be ____________ (convergent/divergent).

Ramanujan probably knew of these series and it occurred to him to look at the *reciprocals of the triangular number* and wonder about them. Was it convergent with its sum growing without end but ever more slowly, and approaching a particular number as a limit? Or, perhaps the sum grows ever more slowly but, like the divergent harmonic series, does not approach any finite limit.

R $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}\ldots$ **K** $\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}\ldots$ **C** $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\ldots$

This new series R, the reciprocals of the triangular numbers (we will call it the Ramanujan series), presents an interesting comparison: It clearly grows more slowly than the harmonic series **C** (compare the trends of the denominators), but only *perhaps* more slowly than the Halfwalk series K.

Answer 2

2a) $1/2^{99}$ (The likely wrong answer is $1/2^{100}$) **b)** 1/100 **c)** Divergent

RAMANUJAN **Unit 47

Now we can speculate that Ramanujan wanted to *show* whether "his" series was convergent or divergent. Probably without help or suggestions from any other mathematician, Ramanujan worked out the essentials of what follows. He started with the reciprocals of the triangular numbers but soon altered their form to make it even easier to change.

***First change:** "Factor out" the 2. If you multiply mentally each term by 2 as indicated below at right, you will see that it gives you back the terms of the left hand expression.
2 x 1/2, 2 x 1/6, 2 x 1/12, etc.

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\ldots = 2 \text{ x } \left(\qquad + \frac{\quad}{\quad} + \frac{\quad}{\quad} + \frac{\quad}{\quad} \quad \ldots \right)$$

1x2 2x3 3x4

Each denominator is the product of two **consecutive** factors. Write these products for *your* answers, too.

$\frac{1}{2}$ $\frac{1}{6}$ $\frac{1}{12}$

***Second change:**
Rewrite each fraction as the difference of two new fractions by extending the patterns.

$$2 \text{ x } \left(\overbrace{\frac{1}{1}-\frac{1}{2}}^{\frac{1}{2}}+\overbrace{\frac{1}{2}-\frac{1}{3}}^{\frac{1}{6}}+\overbrace{\frac{1}{3}-\frac{1}{4}}^{\frac{1}{12}} \quad \_\_\_\_\_ \quad \_\_\_\_\_ \quad \_\_\_\_\_ \ldots \right)$$

4) **Fill in the next three fraction pairs**

See exactly what each bracket encloses and how this affects the total.

***Third change:**
Regroup the fraction pairs.

$$2 \text{ x } \left(\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4} \qquad \ldots \right)$$

5) **Write the next three pairs. Take care with + and -.**

We are at the heart of genius. Ramanujan is saying that no matter how "far out" you extend the series, there will always be a subtracted fraction and then an equal added fraction that eliminate each **other.**

6a) **What value remains now?** ___

b) **So, to what value** is the sum $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}\ldots$ drawing closer and closer? ____

c) This means Ramanujan's series is (convergent/divergent) _______.

7) Also, Ramanujan's series approaches the same limit as the _____________ series.

RAMANUJAN **Unit 47

Answers 3 – 7

3) 1/20, 1/30, 1/42
4x5 5x6 6x7

4) +1/4 – 1/5 , +1/5 – 1/6, +1/6 – 1/7

5) -1/5 + 1/5, -1/6 + 1/6, -1/7 + 1/7 **6a) 2** **b)** 2 **c)** convergent **7)** Halfwalk

---

Ramanujan Recap:

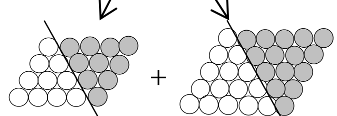
The forming and regrouping into self-eliminating pairs of fractions like -1/4 + 1/4. . ., . . . -1/1000 + 1/1000 etc., is the secret to reducing the entire sum toward 1: It is made possible by denominators like 20, 56, . . . 999000, . . . which are special simply because each contains a pair of factors which are consecutive. Thus, 20 = 4 x 5 allows us to write 1/20 = 1/4 - 1/5, and 999000 = 999 x 1000 which allows us to write 1/999000 = 1/999 – 1/1000.

These come from the remarkable sequence 2, 6, 12, 20, 30, 42… which contains factor pairs which are consecutive numbers such as 12 = 4 x 3 and 30 = 5 x 6. This sequence also appears in the Unit 16: ***More Series,*** following *Tidas and his Crows,* in very different circumstances.

The sums of the even numbers to any point in the series of counting numbers are recorded as 2, 6, 12, etc. Study the information below to see the richness of patterns in these series.

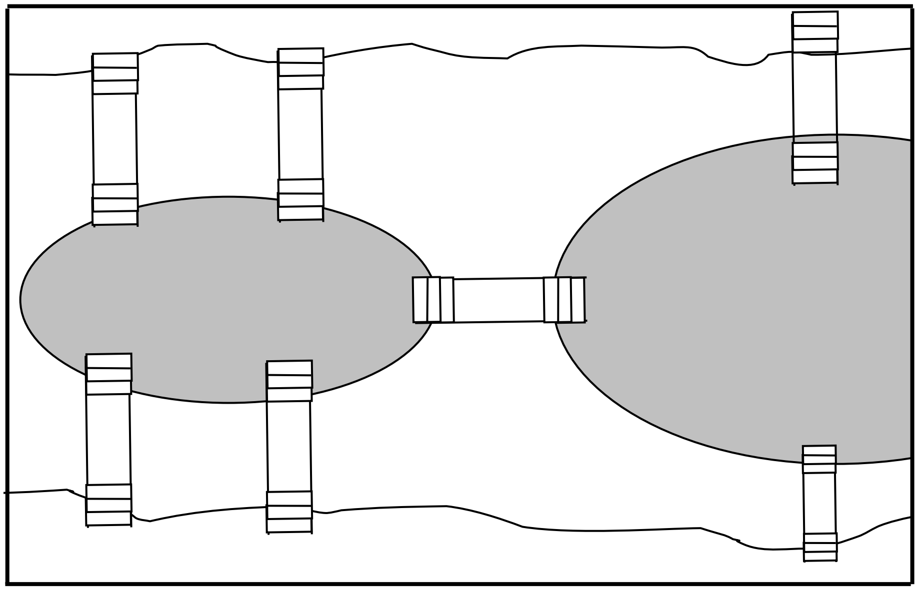
Sums of first n evens, twice the triangular numbers at top. (See page 1 of Unit 16)

1 3 6 10 15 21 28 36
1 2 | 3 4 | 5 6 | 7 8 | 9 10 | 11 12 | 13 14 | 15 16 | . . .
→ 2 | 6 | 12 | 20 | 30 | 42 | 56 | 72 | . . .

. . .  . . .

20 30

THE BRIDGES OF KOENIGSBERG **Unit 48

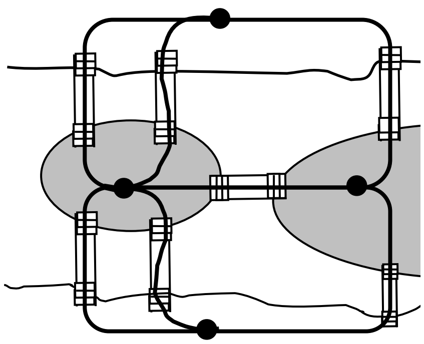


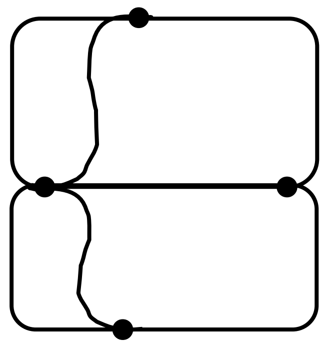
In the late 1700's there was a custom to stroll about on a Sunday in the quiet city of Koenigsberg (pronounced Kernigsberg), Germany. In the city was a set of bridges as shown, and it became a custom to include the crossing of one or more of the bridges as part of the walk. At some point one person mused to another about whether or not it was possible in a stroll including all bridges to walk each bridge exactly once. The question caught the attention of some fellow Koenigsbergians and it became somewhat of a ritual to attempt the walk of all seven bridges without crossing any a second time. Although no person actually achieved the feat (we can speculate that there were claims to the contrary), the practice achieved a certain popularity. In fact, the question reached the notice of the famous Swiss mathematician Leonard Euler (pronounced Oiler, 1707-1783) who stated that the feat could *not* be accomplished, and also found the problem worthy of analysis which has made the question well known even today. (Check the internet.)

Euler's analysis used an important mathematical principle often called *transformation.* In these units we have used the phrase "reducing to a previous case" to express a similar idea, that is, to change aspects of the problem which will allow it to be solved more readily but not change the results. We used the strategy to real advantage in the unit *Carl Friedrich Gauss Teaches the Teacher* and in *To End or Not to End? Is That The Question?*

THE BRIDGES OF KOENIGSBERG **Unit 48

Euler's transformation was to change and extend each bridge to a line and each land mass (island or river bank) to a point. The diagrams show the "shrinkage" of the land masses to points and the transforming of each bridge to a line. The "walking the bridge" problem now becomes a "pencil tracing" problem. From some point (land mass) trace each line (bridge) exactly once with a pencil not leaving the paper or veering off course. Possible?





Euler pointed out that for a finite number of lines and points of this modest size, one could easily enumerate all possibilities and thus demonstrate the proposed "exactly once line tracing" to be impossible.

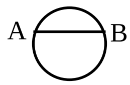
Of course a mathematician of his stature would see immediately the seeds of a larger variety of questions and ideas for investigation. For Euler's own analysis see pages 573 - 580 of *The World of Mathematics Vol. 1*, by James R. Newman, number 13 in our book list. Discussion precedes the analysis which is followed by a chapter on Topology, the now well-known and popular subject invented by Euler as a result of his interest in the bridges, and much expanded by himself and others.

The Moebius (pronounced Merbius) strip is an example of a *topological* surface. It is a simple strip of paper with one twist whose ends are then taped together into a loop with a twist (actually a half-twist). If you try this, trace its complete surface with a pencil and you will find, if you have done it all correctly, that there is only *one surface* to trace. Beyond this is an even greater surprise. With scissors, carefully stab your pencil line and then cut along its entire length without snipping across the loop. Surprised?

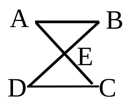
Now we will look at an interesting group of diagrams transformed as the bridge was transformed to lines and points of intersection.

Call the points *vertices* (plural for *vertex*), where line or arc segments arrive. An odd vertex is a point with 1, 3, 5, 7, etc. segments arriving there.

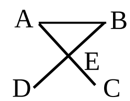
THE BRIDGES OF KOENIGSBERG **Unit 48

Example: 

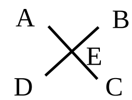
Y | 2 Meaning **Y** for <u>Yes</u>, it is traceable, and, there are **2 odd** vertices.
("Traceable" means every arc traced once without lifting pencil.)

1) 

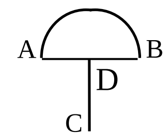
___ | ___

2) 

___ | ___

3) 

___ | ___

4) 

___ | ___

5) Look at exercise 1 and tell at how many vertices you *could* have started tracing successfully ___.

6) Same as question 5 for exercise 2 ___ ; for exercise 3___ ; for exercise 4 ____

Now correct 1 – 6

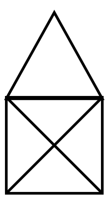
7) You might have drawn some conclusions already. One of Euler's first conclusions concerned odd vertices. In exercise 4, you could have started either at vertex ___ or vertex ___.

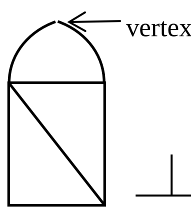
8) But if you started at one of those two vertices, then you had to ________________________.

9) Is your answer to ex. 8 also true for exercise 2? (Y/N) _____

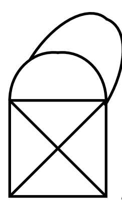
Correct exercises 7 – 9.

Continue answering as you did in exercise 1 – 4.

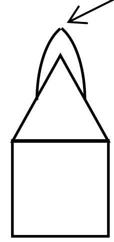
10) 
___ | ___

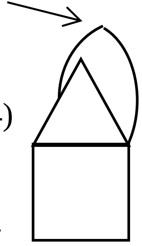
11) 

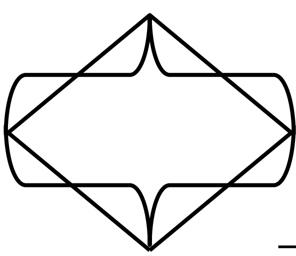
___ | ___

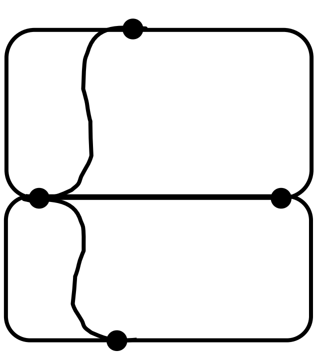
12) 
___ | ___

vertices

13) 
___ | ___

14) 
___ | ___

15) 
___ | ___



16) On a figure with all even vertices (like exercise 15), where can you start to make a successful tracing?____________

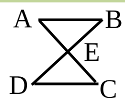
17) Try to state the complete set of conditions which guarantee a successful single tracing.

18) With an odd vertex you must begin or end there. (Do you see this?) This can be done only in the presence of exactly ___ other odd vertex or vertices.

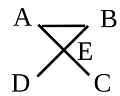
19) Remember the Konigsberg Bridge stroll? There are ___ odd vertices so it can/cannot _____________be traced (strolled without recrossing a bridge).

THE BRIDGES OF KOENIGSBERG **Unit 48

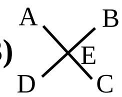
Answers 1 – 19

1) 

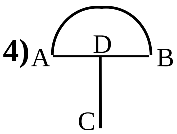
Y | 0

2) 

Y | 2

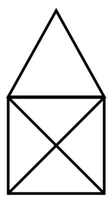
3) 

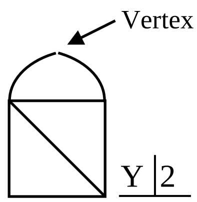
N | 4

4) 

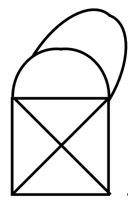
Y | 2

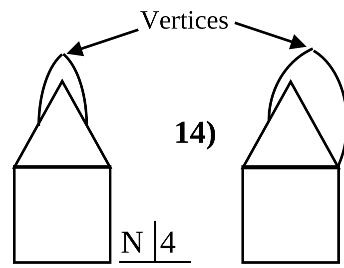
5) 5 **6)** 2, 0, 2, **7)** C or D **8)** Finish at the other **9)** Y

10) 
Y | 2

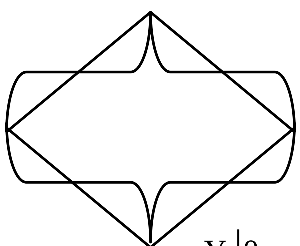
11) 

Y | 2

12) 
N | 4

13) **14)** 

N | 4 Y | 2

15) 
Y | 0

16) If there are 0 odd vertices, *any* vertex can be a starting point.

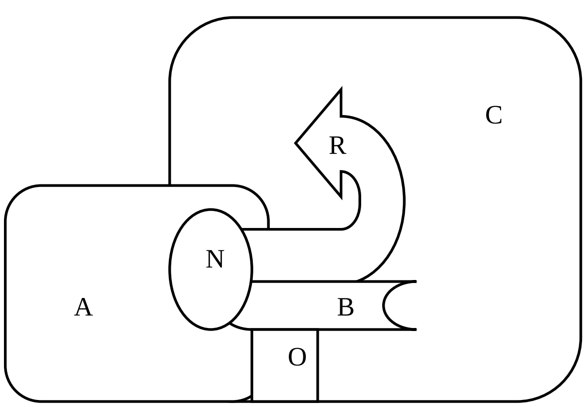
17) For a single tracing of a figure to be possible, there must be exactly 0 or 2 odd vertices.

These figures used to be called networks but modern usage seems to call them graphs since they are part of Graph Theory which is a part of Topology.

18) 1 **19)** 4, cannot

---

20)



A & O, A & C, N & O, N & R, O & R

For a long time there was a famous topology problem called the "Four Color Map Problem".

In any map of "countries" four colors were all that was needed so that two bordering countries would never be the same color. Everyone "knew" that it was true but no one could prove it. Finally, during the 1970's the theorem was proved. Some would not accept it. Others just plain did not like it. The proof was the first of its kind because it required computers to test large numbers of possibilities.

THE BRIDGES OF KOENIGSBERG **Unit 48

Answers 20

In this exercise there are 6 “countries”. They can be colored with 4 colors and still avoid any bordering countries having the same color. Which *single pair* of countries listed could have the same color? (Careful - - only one of the listed country choices is possible) _______

So, did you ever try the Moebius strip mysterious cutting, page 2 of this unit?

If you chose the third answer listed for the four color map problem, investigation would reveal that you could not successfully complete the job. The last pair listed, O & R, is the only pair that will work.

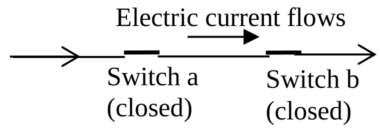
LOGIC AND SYMBOLS ***Unit 49

Inputs a	Inputs b	Outputs a ⋀ b
0	0	0
0	1	0
1	0	0
1	1	1

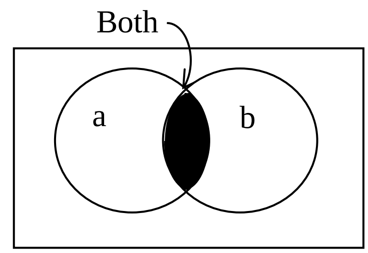
(1) Truth Table

a and b
= a ⋀ b

(2) Conjunction



(3) Circuit Sketch



(4) Venn Diagram of a ⋀ b

In the "truth table" (1), 0 means False and 1 means True. You can see a *binary* counting pattern in the input numerals as you go down the truth table. Study all four diagrams for understanding and come back to this later if not clear now.

1a) T/F: If we had a third input c in the truth table, we would then need another column between b and the double line. ____

b) How many *rows* would then be in the truth table with three inputs? 2^{----} =_____

*c) How many *rows* in a truth table with 5 inputs? 2^{----} = ____

Correct 1a-c now.

The elements of Symbolic Logic, or as now often called, Mathematical Logic, are not numbers; they are statements. Usually a statement is connected to another statement.

a and b
= a⋀b

Let a = Alvin is happy. Let b = Betsy feels blue. Logic is never wishy-washy. A statement is true or it is false; no in between. Strict logic is different from ordinary habits of speech or writing in this respect. There is even a law about it: "The Law of the Excluded Middle". There is no middle between True and False. This allows the building of a truth table as above, using only 0 and 1 as false and true.

d) a ⋀ b = Alvin is happy and ____________________________.

When a statement is put together with another statement as above, we call the connecting word or symbol, not surprisingly, a *connective*.

e) In our truth table it is recorded that a ⋀ b is true only when a is true and _________.
This underlined connective is called a *conjunction*.

f) In diagram (3) we can see that a closed switch allows current to pass through it and in this case, current flow requires both switches to be (closed/open) _________________.

LOGIC AND SYMBOLS ***Unit 49

g) A closed switch has the truth value 1 (true). An open switch, like ___⁄___, has the truth value 0. Current (does/does not) __________ pass through a switch that has truth value 0. (In practice a very small voltage is arranged.)

The existence of *electric currents* with relationships to *truth tables* and to symbols like a$\wedge$b is perhaps our first indication that <u>computers and logic are related</u>.

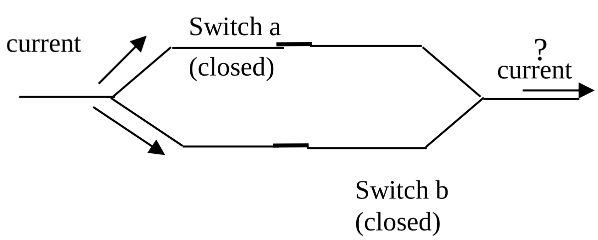
Venn diagrams were developed by John Venn, an English logician (1834 – 1923), and diagram (4) above shows that the statements a and b share a common truth.

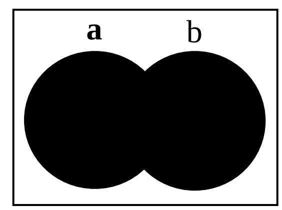
Answers 1a – g:

1a) T **b)** ^{3}, 8 **c)** ^{5}, 32 **d)** Betsy feels blue **e)** b is true **f)** closed **g)** does not

---

a $\vee$ b
= a or b
(or both)
Disjunction





Disjunction

Truth Table

a	b	a $\vee$ b
0	0	0
0	1	1
1	0	
1	1	

2a) **Y/N:** <u>Must</u> both switches be closed for current to flow? ____

The "or" connective, a $\vee$ b, is true when either a is true or b is true or both are true. Compare the switches in the "and" circuit with those in the "or" circuit until you feel that you understand how they work.

b) The third and fourth blanks of the truth table are ____, ____.

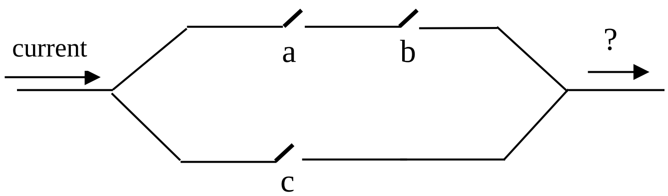
a	~a
1	0
0	1

The next circuit you see will be the simplest of all**:** the "not" circuit. We do not call it a connective because it operates on a single statement. The tilde (~) means "not". So, if **a** is true, ~a is not true. Sometimes a truth table as simple as this one can be confusing. Read the first numeral entry as "when **a** is true then **~a** is false", and:

c) When **a** is false, then ___ is true.

d) **T/F:** ~ ~a is the same as a.___

LOGIC AND SYMBOLS ***Unit 49



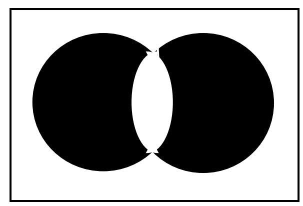
The circuit sketch to the left could be called a disjunction of a switch (**c**) with a conjunction of two switches (**a)** and **(b)**. But you don't have to know the fancy names to understand which switch states (open or closed) allow current to flow.

3 inputs

output, 0 or 1, current flows?

a	b	c	
0	0	0	0
0	0	1	1

3) The truth table to the right has the top row filled with 0's in accordance with all switches open as above. No current will flow. That is the reason that the top *output* is zero. **Fill in** all four columns of the rest of the table. It is usual and sensible that a, b and c of <u>every</u> <u>row</u> be completed first, <u>following the binary counting pattern</u> down to the bottom. **Then** the output column, the truth values of a, b and c, can be determined in turn. Peek at answers on page 4 as little as possible as you work.



4) At right is a new Venn Diagram. If the white oval area were colored black, we would then have a Dis_ _ _ _ _ _ _ _

5a) So, what we have is a disjunction without a con _ _ _ _ _ _ _ _ _.

b) The symbolism for this is **(a ⋁ b) ⋀ ~(_______).** This is often called the "exclusive or" because it means a or b but not both. Circuit designers call it XOR for <u>ex</u>clusive <u>or</u>.

Harriet**'s** mother told her at breakfast that if she straightened up her room she could go to the concert tonight. Harriet sighed but said, "Okay". On her way out the front door to go to school Harriet met her father who had just stepped out to get the morning paper. "Hi, Harriet," he said. "Hi Dad, bye Dad," she said. Her father called after her, "If you don't straighten up your room, I'm afraid you can't go to that concert you've been talking about all week." "Dad, Mom just told me that!" "Well," said Dad, "I don't think she said exactly that."

"No, different words but the same idea," Harriet responded, a little irked by now.

"Nope," said her father, "Not exactly the same idea, either."

"What's the difference?" demanded Harriet, and then, "Never mind, never mind. I have to go."

LOGIC AND SYMBOLS ***Unit 49

The exchanges between Harriet and her mother and father bring up perhaps the most interesting connective in logic. This connective is also in need of attentive patience. This is the "if - - - then" connective and its various forms and close relatives. Let **s** represent the statement "Harriet straightens up her room" and let **c** represent "Harriet goes to the concert".

6) Mom's condition is **s** $\longrightarrow$ **c** **(**if **s** then **c**)**.** Dad's condition is ~**s** $\longrightarrow$ ____ **;** that is, **if <u>not</u> s**, **then not c.**

It is clear that the *forms* of the if - - - then statements are different, but does this reflect merely a difference in words as Harriet thought or a difference also in meaning as her dad thought. Here again is a case where strict logic does not agree with the everyday conversation meaning. The connective $\longrightarrow$, is called an "implication".

s $\longrightarrow$ **c** says "**s** implies **c".** In daily use, "imply" means to *suggest*, while in logic, imply is a *must*.

s $\longrightarrow$ **c** means the statement **s** is necessarily and always accompanied by the statement **c.** But this does not solve the problem of whether there is really a *meaning* difference between **s** $\longrightarrow$ **c** **and** ~**s** $\longrightarrow$ **~c,** Mom's and Dad's statements respectively**.** Later**.**

Answers 2 – 6

2a) No **b)** 1, 1 **c)** ~a **d)** T

3)

a	b	c	output (current?)
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

4) disjunction. **5a)** conjunction (or "and"). **b)** a $\wedge$ b. **6)** ~**c.**

LOGIC AND SYMBOLS ***Unit 49

Most people would agree with Harriet that her mother truly meant that she *must* straighten her room before going to the concert. But, her mother did not really say that.

If Harriet were adventurous, she might deliberately avoid the mess in her room and then have a lively argument with her mother about whether she could go to the concert. She would probably lose with her mother but *certainly* would lose with her father's argument: **~s** $\longrightarrow$ **~ c (**If Harriet does <u>not</u> straighten her room she <u>cannot go</u> to the concert).

Look at the truth table and see how many of the output blanks you can fill with confidence. You might find trouble being confident in two or three cases. The bottom output is 1. The others might be doubtful but if you think carefully you can decide about the third. It says that **s** is true and **c** is false.

(outputs)

s	c	s $\rightarrow$ c
0	0	
0	1	
1	0	
1	1	

7a) Use **s, c,** $\longrightarrow$ and ~ to make that same statement "If **s** is true then **c** is false**".** __________

The implication itself declares if **s** (is true) then **c** (is false**).** But a **true s** *requires* a **t**rue **c** or else Mom's statement is contradicted. So fill in the third output.

We have not yet dealt adequately with the first two rows of the truth table. Those rows have in common that **s** is not true. It might seem reasonable to say that if **s** is false, we do not have any information about **c;** we are told only that if **s** is *true* then ***c*** is true.

We might then argue that if **s** is false (0), **c** could be whatever we choose, 0 or 1. Yes, but we will get better information on why the first two outputs are 1 and 1 from exercise 9 later.

b) **s** (true) is sufficient to know **c** (is true). T/F ____.

c) Therefore, when **s** is true, **c** must not be false. T/F ____ .

d) While Mom's condition is *sufficient* for permission; if **s**, then Harriet may go, Dad's condition is *necessary; <u>only</u> if **s*** may Harriet go. ____

e) The question 7c, deals with the (second/third) __________ row of the table.

Answer 7

7a) s $\longrightarrow$~c **b)** T **c)** T **d)** T **e)** third

LOGIC AND SYMBOLS ***Unit 49

In the truth table for **s, c, and s $\longrightarrow$c,** above, the *top two* **s$\longrightarrow$c** values, are 1 and 1; the third is 0 the and fourth is 1.

It took two parents to do it, but Harriet is securely bound by a sufficient ($\longrightarrow$) and necessary ($\longleftarrow$) pair of conditions, ($\longleftrightarrow$), Mom's and Dad's, respectively.

If **s** then **c,** *and* **s** only if **c**. Both together make **s$\longleftrightarrow$ c.** $\longleftrightarrow$ means is equivalent to.

Harriet has permission for the concert $\underbrace{\text{if and only if}}_{\text{iff}}$ she straightens up her room.

8) Exercise 7c on the previous page said, in effect, that if we have s $\longrightarrow$ c, then we cannot have **s** true without having **c** true. That is, we cannot have **s$\wedge$~c. Not** having **(s$\wedge$~c)** is equivalent to saying **~(s______**) .

9) **If (s$\wedge$~c)** is false then ~(**s**$\wedge$~**c**) is **T/F**___.

1 s	2 c	3 ~c	4 (s $\wedge$~c)	5 ~(s $\wedge$ ~c	6 (s $\longrightarrow$c)

Patience may be needed for columns 4-6.

10) Fill in the left two columns of the truth table starting downward with our binary counting: 0 0, 0 1, etc. Then do the third column by inspecting column 2. Easy. Next do the fourth column, the conjunction of the first and third columns. Note: 0 $\wedge$ 0 = 0. After that the fifth column is easy.

Recall that we already devised the connective for the fourth column simply by realizing that having **s and ~c** was the *single false case* for **"if s then c".** That fact can give you columns 4, 5 & 6.

Check the answers to exercises 8 – 10 now.

LOGIC AND SYMBOLS ***Unit 49

11) A compound statement: "If Harriet's older brother Michael is **g**rumpy, then Harriet **k**eeps her distance". Answer **T, F, or S,** with **S** meaning "**S**ometimes**",** depending upon this particular information.

a) Write the above statement symbolically using k for Harriet keeps her distance and g for Michael is grumpy). _________

b) ~g $\rightarrow$ ~k is the *inverse* of g $\rightarrow$ k. Is ~g $\rightarrow$ ~k T, S, or F? _____

c) Instead of g $\rightarrow$ k, consider k $\rightarrow$ g, the *converse* of g $\rightarrow$ k. Write its translation into English and answer T, S, or F. _____

d) Finally, we can have a *contrapositive* of **g $\rightarrow$ k.** This is **~k $\rightarrow$ ~g.**

Experiment with this and tell what definite conclusions you find about the contrapositive.

____

____

Answers 8 – 11

8) **$\wedge$~c**

9) T

10)

1 s	2 c	3 ~c	4 (s $\wedge$~c)	5 ~(s $\wedge$ ~c	6 (s $\rightarrow$c)
0	0	0	0	1	1
0	1	0	0	1	1
1	0	1	1	0	0
1	1	0	0	1	1

The most significant outcome at right is that columns 5 and 6 are identical, confirming the statement that you can*not* have **(s $\rightarrow$ c)** with **s** true and **c** false; that is, having **s** means you *must* have **c,** because you can*not* have **s and not c** (column 5). Column 4 makes column 5 easier to get right. Those two truth table sets are complementary (not compl**i**mentary).

11a) g $\rightarrow$ k **b)** S **c)** S, depends on what is said.

d) T. The contrapositive of a true implication is always true. (Such a truth is called a *tautology* – true by virtue of its logical form.) (*Tautology:* Webster's 9th Collegiate Dictionary)

LOGIC AND SYMBOLS ***Unit 49

Comparing individual columns of truth tables is a way of proving equivalence.

12) Here is a long, surprising, but not difficult truth table. Columns 7 and 8 are easy because of 5 and 6. Columns 1 – 4 should be *really* easy.

1	2	3	4	5	6	7	8	9	10
a	**b**	**~a**	**~b**	**a ⋀ b**	**a ⋁ b**	**~(a ⋀ b)**	**~(a ⋁ b)**	**~a ⋁ ~b**	**~a ⋀ ~b**
				0	**0**				
0	**1**			**0**	**1**				
				0	**1**				
				1	**1**				

Correct the above table. Be aware of patterns in the answers as you go.

Column 8 is the "not", or complement, of column 6. Likewise with columns 7 and 5.

13) Column 8 also has a relationship to column 10. They are _________________.

14) Find another pair of columns with the same relationship. _________________.

Take careful note of the connectives, including the "nots", in those equivalent pairs.

15) Finish the actual equivalence of connectives for columns 8 & 10: ~(a ⋁ b) ⟷ ____________

16) Do the same for the columns 7 & 9. ___________ ⟷ ____________

DeMorgan's law (1),

17) Complete: The "not" of the conjunction of two statements is equivalent to the disjunction of ______________________________________.

DeMorgan's law (2)

18) Similarly, state the relationship of column 8 to column 10.________________________
____

19) Also, state the relationship shown by the connectives in 6 & 8. ____________________

20) Is the relationship in exercise 18 always true? __________

LOGIC AND SYMBOLS ***Unit 49

Answers 12 – 20

12)

1	2	3	4	5	6	7	8	9	10
a	b	~a	~b	$a \wedge b$	$a \vee b$	$\sim(a \wedge b)$	$\sim(a \vee b)$	$\sim a \vee \sim b$	$\sim a \wedge \sim b$
0	0	1	1	0	0	1	1	1	1
0	1	1	0	0	1	1	0	1	0
1	0	0	1	0	1	1	0	1	0
1	1	0	0	1	1	0	0	0	0

13) equivalent **14)** Columns 7 & 9 **15)** $\sim a \wedge \sim b$ **16)** $\sim(a \wedge b)$, $(\sim a \vee \sim b)$

17) the "not" of each statement.

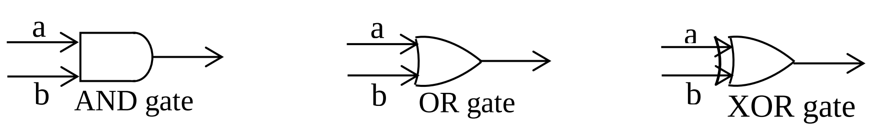
18) The "not" of the disjunction of two statements is equivalent to the conjunction of the "nots" of the statements. **19)** One is the "not" of the other. **20)** Yes

---

Symbolic Logic is used extensively in the circuit designs of computers, calculators, cell phones and countless other electronic applications in business, schools and homes.

Those who design these thousands of applications arrange the routing of electronic messages through incredible mazes of tiny chips, diodes, integrated circuits, resistors, etc. These designers use electronic *gates* (corresponding with symbolic logic symbols) as their *deciders* in determining what message will be sent to the next component. Typically, the only message a gate can send is one bit, 0 or a 1.

The only message a gate can receive is a *pair* of bits: 00, 01, 10, or 11. The 1 indicates a closed "switch" allowing 5 volts of current, while a zero switch is "open" and admits only a very small current.



LOGIC AND SYMBOLS ***Unit 49

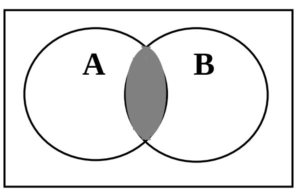
The output from a gate is one bit, 0 or 1. The input to a gate is two bits, with each 0 or 1. In your truth table work you have seen inputs called a and b, rather than 0 or 1. This is so we can speak definitely about bits whose value is not necessarily known. A gate is like an electronic truth table. The a and b inputs are 00, 01, 10, and 11.

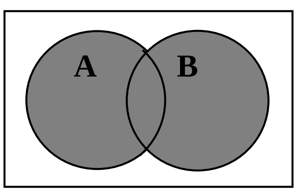
We saw the corresponding outputs as patterns of bits, 0001 for AND, and 0111 for OR. Any new connective, or gate, would also have a four bit pattern: $\square\square\square\square$ **.** The pattern for the new XOR gate is 0110 and you will soon see why. The question now is, "How many different patterns of 0's and 1's could exist in four spaces?". We know of three patterns appearing above.

21) There are 4 spaces and 2 ways to fill each, so: __ x__x__x__ = $\_\_\_^{\_}$ = ____

Check your answer to exercise 21 now.

Here we mention in passing the close connection of Symbolic Logic to the Algebra of Sets and to Boolean Algebra.





Set A = {1, 2, 3, 4, 5} **Set B** = {4, 5, 6, 7, 8}

22) The *Intersection* of these Sets A, B is A *and* B and contains which numbers from above? $A \cap B$ = {___, ___)

23) The *Union* of Sets A, B is A *or* B =

$A \cup B$ = {__, __, __, __, __, __, __, __}

Boolean Algebra is a *two valued* algebra such as 1,0 or is, is not, as opposed to the algebra of the real numbers that has $2^{\aleph_0}$ values, according to Georg Cantor.

$\overline{A}$ = Not A. $A \cdot B$ = A *and* B. $A + B$ = A *or* B

In Boolean Algebra, DeMorgan's Law (1) says: $\overline{A \cdot B} = \overline{A} + \overline{B}$. (It *might* be helpful to compare on page 9 ex. 12, columns 7 and 9.)

Answers to 21-24 are on the next page.

24) Similarly, DeMorgan's Law (2) says: $\overline{A + B}$ = _____.

25) You found in ex. 21 that there are 16 possible output patterns in a two-value algebra. How many Venn diagrams are on page 11? _____

LOGIC AND SYMBOLS ***Unit 49

26) Notice the binary numbering of the Venn diagrams in Row 1. Continue the numbering though 1111, which equals 15, and then check answers. **Note:** (0 to 15 = 1 to 16)

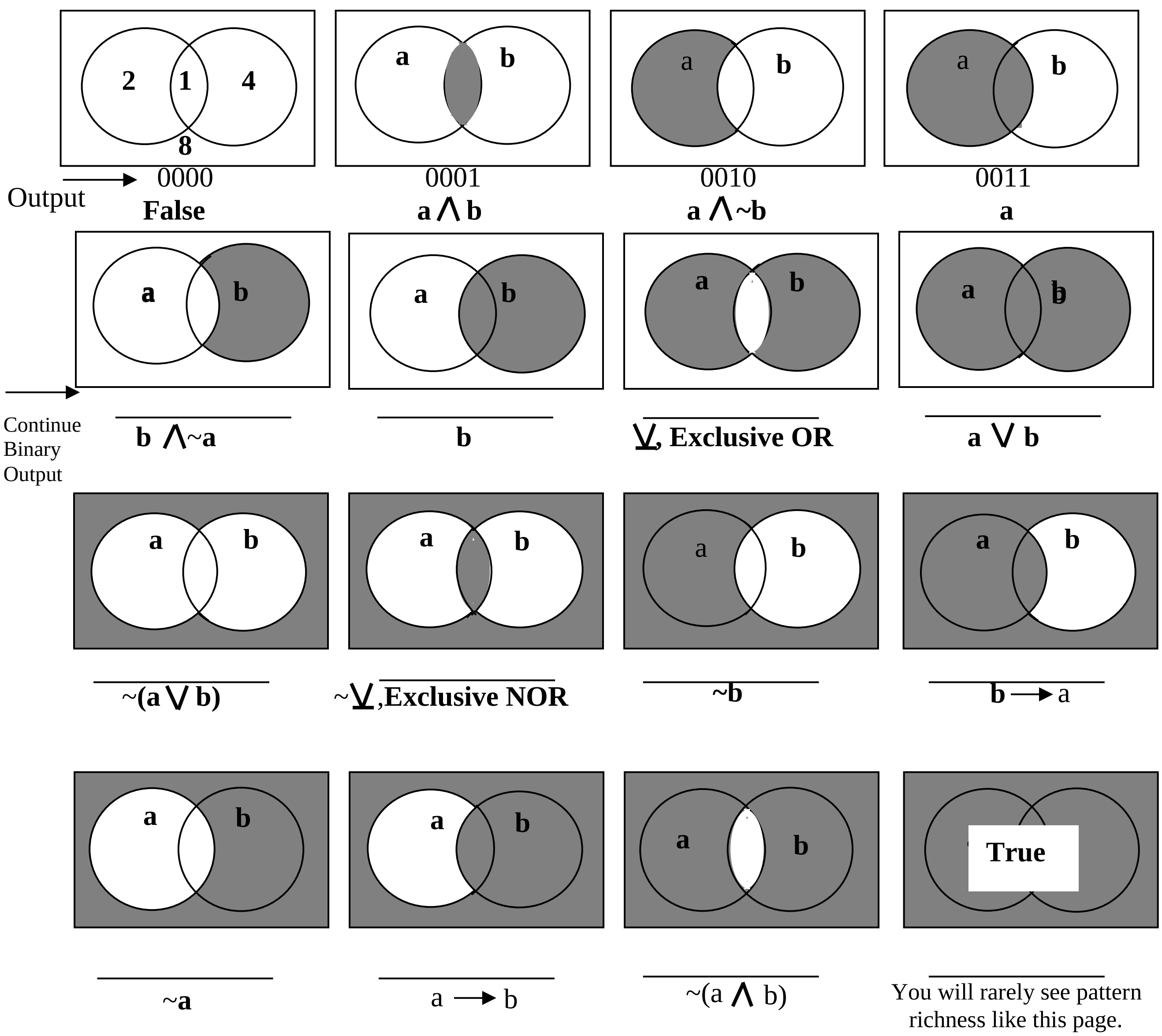
27) How many distinct, separate spaces are available for shading in a Venn diagram? _______

28) Compare the shading in all of row 3 with all of row 1. ___________________________

29) **Y/N:** Does row 4 compare with row 2 in the same way as row 3 to row 1? ______

Answers 21 – 24

21) 2x2x2x2=2^4=16. **22)** 4, 5 **23)**1, 2, 3, 4, 5, 6, 7, 8 **24)** $\overline{A} \cdot \overline{B}$



Logic and Symbols ***Unit 49

30) Is it likely that John Venn *intended* exactly 4 spaces to combine for shading? _____

31) Is every possible shading combination shown on this page? ____

32) The second diagram shading is opposite to that of the Row 4, Column ___ diagram.

33) The same is true for Row 2, Column 3 compared to Row ___, Column ___.

34) Contemplate Row 3, Column 4. Which diagram is its opposite? Row ___Column ___

35) Refer to the space values assigned in the first diagram. Note that in diagram 2 the space valued 1 is shaded, and the binary numeral 0001 is assigned. Check some others. Does each shading reflect exactly the binary number of its diagram? _____ Also, are the binary numerals the appropriate truth table outputs? (See $\wedge$ and $\vee$)? _____

*36) Confirm that shading, numerals *and names* **reflect** their opposites; first and last, second and second from last, third and… Are there any exceptions to this? ______

Answers 25 – 36

25) 16

26) 0100, 0101, 0110, 0111,- 1000, 1001, 1010, 1011, -- 1100, 1101, 1110, 1111

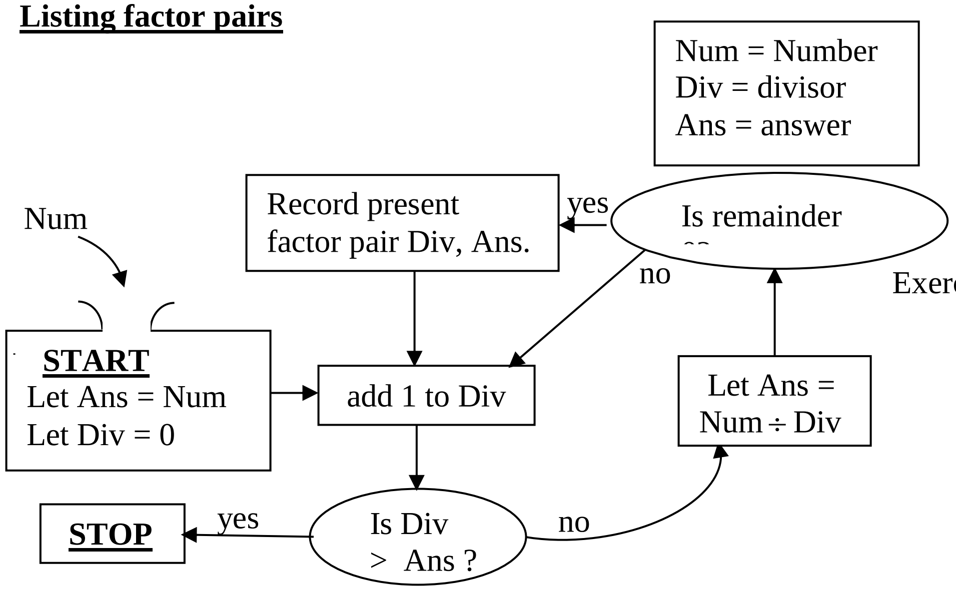
27) 4 **28)** Shading is the same except that backgrounds are opposite.

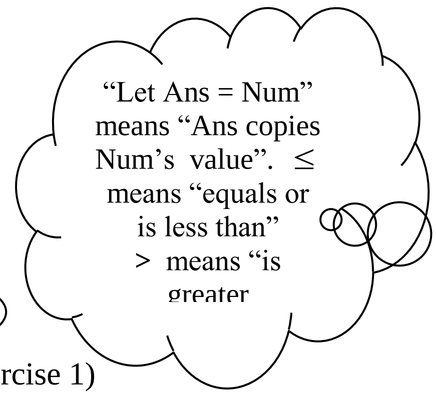
29) Yes **30)** Yes **31)** Yes **32)** 3

33) 3, 2 **34)** 2,1 **35)** Yes, Yes **36)** No

FLOW CHARTS *Unit 50

Listing factor pairs





Exercise 1)

Factor Pairs
for Num = 63

Div	Ans
1	63
3	___
___	___

2)

Factor Pairs for Num = 48	Div →						
	Ans →						

Div →						
Ans						

3)

Factor Pairs for Num = 64	Div →						
	Ans →						

4) If you were asked to list the factor pairs for 100, what would be the value of Div which would stop the run? ______

*5) Same question as 4) for 102? _____; for 111? _____; for 122? _____; for 119? _____

6) To find all factors of a number, this program would test to the whole number just above its __________ __________

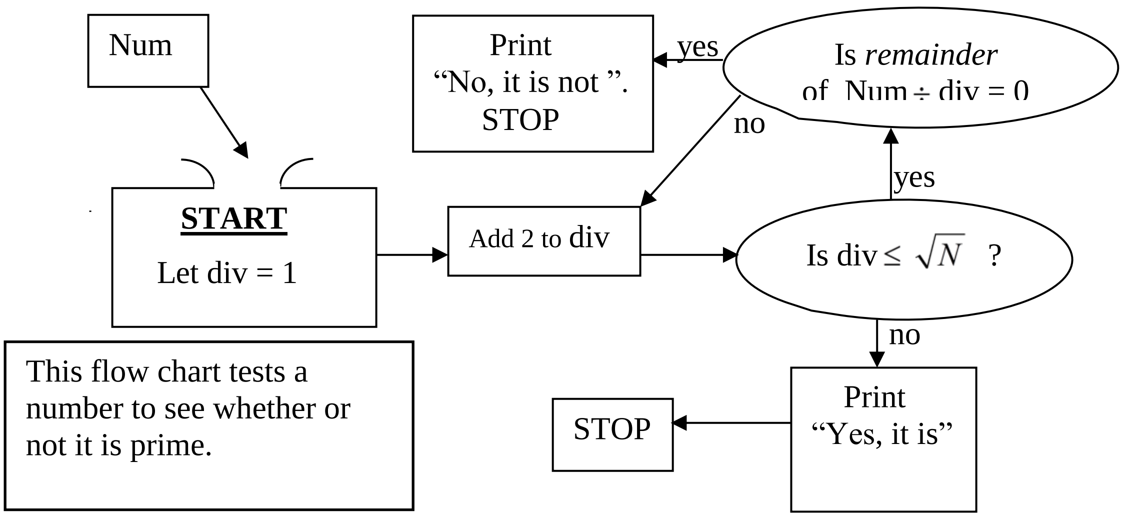
*7) If you could transform into a computer language the steps in the factor listing flow chart (this probably would not be hard to do), what would be the last factor pair out of the many pairs that it would print, for:

a) 64 ______ b) 169,000,000? ______________________ c) 1200 ______________

d) 25 x 10^{40}? ___________________________________

(Use an exponent in your answer)

8) Someone without the experience of this paper or something similar, if asked to list all the factors of 400 (not necessarily in pairs), would probably go how high to find them all? __________ You, of course, would know you have them all by the time you got to _____.



9) To test whether the flow chart works correctly, mentally run thru the chart with each of the following numbers and tell whether the chart's answer is "Yes" or "No" for each number that you try. You might not need the chart but if you don't use the chart you won't be testing *it*.

a) 69 _______ b) 49 _______ c) 79 _______ 39 _______

10) Explain why $\sqrt{N}$ is a stopping place.____

____

Note: Such flow charts are used less often now because modern computer languages have many routine procedures built into their code. Flow charts have been around for much longer than computers and are used to organize complex planning for such things as business meetings and large company staff assignments.

For our purposes flowcharts display the procedure and its repeated "looping". Also, the making of decisions and their strategic placement is clear.

FLOW CHARTS *Unit 50

Answers

1)

Factor Pairs
for Num = 63

Div	Ans
1	63
3	**21**
7	**9**

2)

Factor Pairs for Num = 48	Div →	1	2	3	4	6	
	Ans →	48	24	16	12	8	

3)

Factor Pairs for Num = 64	Div →	1	2	4	8		
	Ans →	64	32	16	8		

4) 11 **5)** 11, 12, 13, 12 **6)** Square Root

7a) 8, 8 **b)** 13000, 13000 **c)** 30, 40 **d)** 5×10^{20}, 5×10^{20}

8) Probably 200, halfway; 20, the square root of 400.

9a) No **b)** No **c)** Yes **d)** No

10) To go higher than $\sqrt{N}$ would find factors already known.

PERMUTATIONS 1 *Unit 51

(No Calculator)

Sometimes it is useful in mathematics to know how many ways a group of objects can be arranged. For example, how many arrangements can be make with the letters a, b, and c? Writing them out, we have:

abc	bac	cab
acb	bca	cba

These arrangements are called <u>permutations</u>. It is clear that there are six permutations of any 3 objects.

1) Cover the above abc's and write all possible permutations of the letters x, y, and z.
2) Write all possible permutations of the numbers 1, 2, and 3.
3) If Jack, Jim, and Don have 3 seats reserved at a Saturday football game, in how many ways could the three boys be seated? ______
4) List all the possible arrangements of the letters a, b, c, and d. It will be easier to keep track of them if you use some kind of system, such as writing all possible arrangements with a as the first letter and then with b as the first letter, etc. An answer is given at the top of page 2, but of course your system might be different.

> Suggested start, but a different one is O.K. if you can keep it organized in some way so you don't lose track.

a b c d	b a c d	______	______
ab_ _	b a _ _	______	______
a c _ _	______	______	______
_ _ d b	______	______	______
a _ b c	______	______	______
_ _ _ _	______	______	______

5) Using the information from example 4, how many different ways could four girls be seated in their four seats at a basketball game? _____

PERMUTATIONS 1 | *Unit 51

Answers 1 – 5

1) xyz yxz zxy
xzy yzx zyx

2) 123 213 312
132 231 321

3) 6

4)

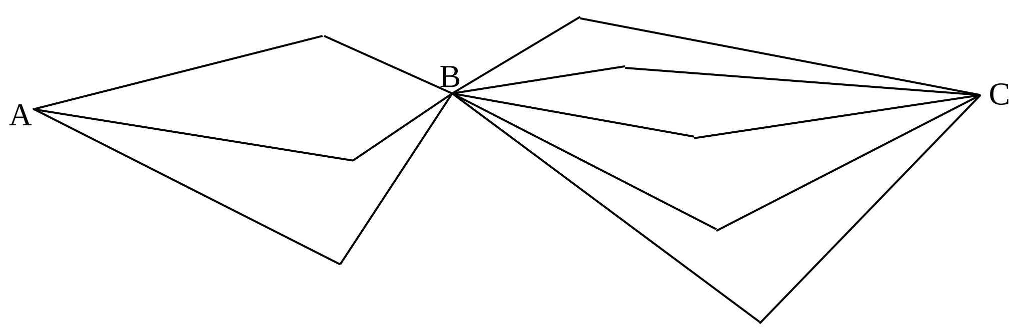
abcd	bacd	cabd	dabc
abdc	badc	cadb	dacb
acbd	bcad	cbad	dbac
acdb	bcda	cbda	dbca
adbc	bdac	cdab	dcab
adcb	bdca	cdba	dcba

5) 24

How many possible arrangements could be made using eight objects? What a job it would be to write all these out! There is a mathematical way of finding this out which is not at all difficult.

Suppose there are three possible routes from Town A to Town B, and 5 possible routes from Town B to Town C. How many different routes can you travel from Town A, through Town B, to Town C?

The situation looks like this.



If we take the upper route from A to B there are five possible ways to complete our journey to C. If we take the middle route, or the lower route to B, there are five ways to complete the journey for each of these routes. Therefore there are 3 x 5, or 15 ways to travel from A to C. Did you figure it this way? The point is, if you can do one thing in 3 ways and another thing in 5 ways, you can do one thing and then the other in 15 ways.

6) If there are 4 highway routes from Boston to New York and 4 from New York to Chicago, how many highway routes are there Boston to Chicago? ______

7) If there are 3 routes from A to B, 6 routes from B to C, and 4 routes from C to D, how many possible routes are there from A to D? _____

8) If Mrs. Fleespot has a choice of 6 colors to paint her shingles, and 8 colors to paint her trim, how many color combinations does she have from which to choose? ____

9) Clair Shuman has seen her first name spelled “Clair, “Clare”, and “Claire”. She has seen her last name spelled “Shuman”, “Schuman”, “Schumann” and “Shumann”. Using these possibilities, how many different ways could her whole name be spelled? ____

10) If event A can happen in 5 ways and event B can happen in 7 ways, in how many ways can the event A followed by B happens? ____

11) Don has 4 pairs of shoes, 6 pairs of pants, and 3 jackets. How many combinations of shoes, pants and jackets can he wear? ____

12) If event A can happen in 4 ways , event B in 3 ways and event C in 2 ways, in how many ways can A followed by B followed by C happen?____

Returning to the problem of finding the <u>number</u> of *arrangements* eight objects can have:

Suppose the objects are the letters a, b, c, d, e, f, g, h.

How many choices do we have for writing the first letter? (Place your answer in the first block above.) Remember that your answer is a number, not a letter.

After we have counted one letter to use the first space, how many possibilities remain for selecting the second letter? (Place your answer in the second block.)

How many remain for the third choice, the fourth, etc.? Fill in all the spaces.

The number of arrangements, then, is 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1.

13) When multiplied out, this product gives ________ (You do it)

14) Using this method, find the number of permutations of 4 objects. ____

PERMUTATIONS 1 *Unit 51

15) The first row in the math class has 6 people. In how many ways can they be arranged?____

16) How many different arrangements can be made of the letters in the word "company"?____

17) How many different 5 digit numbers can be made of the digits 3, 4, 7, 8 and 9, if each digit is used only once in a number? ______

18) The first row of a jury consists of 6 people. If the jury foreman must sit on the left end, in how many ways can the first row be arranged? _____

*19) How many ways could the front row be arranged (in problem 18) if the foreman may sit on either end? ____

20) Find the number of possible arrangements of 7 hikers walking single file. _______

No calculator.

The expression 5 x 4 x 3 x 2 x 1 is called *5 factorial*. The mathematical symbol for 5 factorial is 5! Likewise, 4! = 4 x 3 x 2 x 1.

21) 3! = ____ 22) 9! = __x__x__x__x__x__x__x__x__ = __________

23) Use ex. 22 to shorten the work for finding 11! _____________

24) "Toughie"

Since 20! = 20 x 19 x 18 x . . . x 3 x 2 x 1, does n! = n x (n – 1)x(n – 2)x(n – 3) . . . . 3 x 2 x 1? (yes/No) _____ This is usually written n(n – 1)(n – 2)(n – 3) . . . 3 x 2 x 1.

25) Write a formula for the number of possible permutations of n objects. Let P stand for permutations. _______

26) $\frac{8!}{5!}$ = _____

27) $\frac{10!}{5! \times 4!}$ = _____
Write it out

28) $\frac{6! \times 3!}{8!}$ = _____

29) Toughie! How many different four-digit numbers can be made using the digits 0, 1, 3 and 4 if digits are not repeated in a number? Remember that 0 cannot come first. ________

30) $\frac{123!}{121!}$ = ________

!!!

PERMUTATIONS 1 *Unit 51

Answers 5 – 28

6) 16 **7)** 72 **8)** 48 **9)** 12 **10)** 35 **11)** 72

12) 24 **13)** 40,320 **14)** 24 **15)** 720 **16)** 5040

17) 120 **18)** 120 **19)** 240 **20)** 5040 **1)** 6 **22)** 362,880

23) 39,916,800 **24)** Yes **25)** P = n! **26)** 336 **27)** 1260

28) 3/28 **29)** 18 (3x3x2x1) **30)** 15,006

PERMUTATIONS 2 *Unit 52

In part A of this paper we discovered how to find the number of arrangements, or permutations, of any number of objects. Thus 5 objects could be arranged in 5! ways. This, we found out, means 5 x 4 x 3 x 2 x 1, or 120 ways. In general, n objects can be arranged n! ways if all the objects are used. But what happens if we don't use all the objects in making arrangements? Suppose that we want to know how many different 3 digit numbers we can get using the digits 3, 5, 7, 8, 9, without repeating digits in the same number. Remember that we are using only three digits at a time.

1) For the first digit of our three digit number, how many choices do we have?____
2) After using one digit for the first digit in the number, how many choices do we have left for the second digit? ____
3) How many possibilities remain for the third digit? ____
   Therefore, the number of arrangements of 5 digits, taken 3 at a time, is 5 x 4 x 3, or 60 arrangements.
4) How many permutations can be made of 6 letters taken 4 at a time? ____
5) How many 5 digit numbers can be made using the digits 1 through 9 without repeating digits in the same number? ____________
6) List all the permutations of the following five symbols taken two at a time +, x, -, /, = remembering that +, - and -, +, are two different permutations. Start with a column of 4 pairs each beginning with +.
7) How many arrangements of the symbols in ex. 6 can be make taken four at a time? (Don't list them) ______

*8) How many arrangements of 100 objects can be make taking a pair at a time? _____

9) Mrs. Mason took her five troublesome sons to New York on the train. It was necessary for three boys to sit together on one seat. She tried to find the best arrangement possible for placing 3 of the 5 boys together for a minimum of confusion. How many arrangements could she possibly try? _______ (Poor Mrs. Mason!)
10) Allowing no repetitions of digits in a number, how many 4 digit numbers can be made using all digits? (Remember that 0 cannot come first.) ________
11) Allowing no repetitions of letters in a group, how many four letter arrangements can be made using the entire alphabet? __________________

PERMUTATIONS 2 *Unit 52

So far, we have been careful not to allow repetitions of objects within a group. Let's see what happens when we do allow repetitions.

Illustration:

- How many different permutations of the letters a, b, c, can be made allowing repetitions?
- There are three possible choices for the first letter.
- There are also three possible choices for the second letter since the first choice could be repeated. Likewise, there are three possible choices for the 3rd letter.

12) Therefore, allowing repetitions, there are how many possible arrangements of a, b, c? _____

13) List all the permutations of exercise 12. Be sure to use a system. It's not easy. Below is a start. We will specify repetitions or replacements when they are to be used.

    aaa
    aab
    aac
    aba

14) How many four digit numbers can be made from the digits 2, 3, 5, 8, 9, allowing repetitions? _______

15) There are four marbles in a bag; one black, one blue, one brown, and one white.
    a) In how many ways can you draw 4 marbles from the bag, one after another?____
    b) In how many ways can you draw four marbles from the bag if each marble is replaced immediately after it is drawn? (Of course, in this case, it could be drawn again.) ______
    c) In how many ways can you draw three marbles without replacement? _____
    d) In how many ways can you draw two marbles with replacement? _____

16) How many six digit number plates can be made using the digits 1 through 9 allowing repetitions? _________________

17) On a certain combination lock the dial is marked off from 1 to 50. How many 3 digit combinations are possible for such a lock?
    a) Allowing repetitions ___________
    b) Not allowing repetitions ___________

PERMUTATIONS 2 *Unit 52

18) In how many ways could you draw three cards from a full pack if a card is not replaced when drawn? _______________

19) How many arrangements of five objects can be made using all objects in each arrangement and allowing repetitions? ______________

20) A packet of colored paper contains 6 sheets, each of a different color. In how many ways could you draw 3 sheets from the packet?

   a) with replacement ________

   b) without replacement _________

Answers

1) 5 **2)** 4 **3)** 3 **4)** 360 **5)** 15,120

6)

+ x	x +	- +	= +	● +
+ -	x -	- x	= x	● x
+ ●	x ●	- ●	= -	● -
+ =	x =	- =	= ●	● =

7) 120 **8)** 9900 **9)** 60 **10)** 4536 **11)** 358,800 **12)** 27

13)

a a a	b a a	c a a
a a b	b a b	c a b
a a c	b a c	c a c
a b a	b b a	c b a
a b b	b b b	c b b
a b c	b b c	c b c
a c a	b c a	c c a
a c b	b c b	c c b
a c c	b c c	c c c

14) 625 **15a)** 24 **b)** 256 **c)** 24 **d)** 16

16) 531,441 **17a)** 125,000 **b)** 117,600

18) 132,600 **19)** 3,125 **20a)** 216 **b)** 120

PERMUTATIONS TOUGHIES **Unit 53

Review Summary

A. The number of permutations of n things is n! (n factorial).

B. The number of permutations of six things taken four at a time is: 6 x 5 x 4 x 3, or 360

C. The number of permutations of four objects, allowing repetitions, is: 4 x 4 x 4 x 4 = 256

D. The number of permutations of six objects taken four at a time and allowing repetitions is: 6 x 6 x 6 x 6 = 6^4 = 1296

An easy one for warm-up:

1) How many arrangements can be made with seven letters?

   a) Using all letters in each arrangement? ________

   b) Taken four at a time? _______

   c) Using three letters at a time, allowing repetitions? _______

Now for the toughies: (Reminder: There are 26 letters in the English alphabet.)

2) How many different six-digit car registration plates can be made?

   a) Using any letter except o or i as the first digit and any numerical digit for each of the last five digits? (Note: A number like B03365 is O.K.)______________________

   b) Using any letter except o or i (do you see why?) as the first and last digits and any numerical digit as each of the four middle digits? (A number like B0013M is O.K.) __________________

*3) How many possible permutations of the letters a, b, c, d are there if we can have anywhere from 1 to 4 letters in a permutation, not allowing repetitions? ___________

   Hint: Add the numbers of arrangements of 4, 3, 2 and 1 letters.

4) How many different arrangements can be made using each of the letters in the word "bread" only once? ______

PERMUTATIONS TOUGHIES **Unit 53

If not specified, assume that all elements are used and no replacements are made.

*5) How many different arrangements can be made using all of the letters in the word "computer" ______________

*6) To the nearest day, how many days would it take to make all possible arrangements of ten books on a shelf if 10 seconds are required for each arrangement? (Assume a working day of 24 hours.) ____________

**7a) In many states you may have your car number plate composed of letters to form your initials or a short nickname for an extra charge. If the plate can be composed of one, two, or three letters, how many plates could be made? (Repetitions are allowed.)______________________

**7b) Just as above except that you may have four, three, or two letters, no 1-letter plates and no blank ones. ________

8) Allowing repetitions, 5 objects may be arranged in 5 x 5 x 5 x 5 x 5 ways. Write this expression using an exponent. ____

9) Allowing repetitions, 24 objects may be arranged in how many ways? Use an exponent; don't work it out. _______

10) Use a general expression containing an exponent to indicate how many ways, allowing repetitions, that n objects could be arranged._____

11) Use a general expression to indicate how many ways, allowing repetitions, that n objects could be arranged taken 4 at a time ______

*12. Use a general expression to indicate how many ways, allowing repetitions, that n things could be arranged taken n at a time. ____

*13. Use a general expression to indicate how many ways, <u>not</u> allowing repetitions, that n things could be taken 4 at a time. Let (n – 1) mean 1 less than n. ________________________

PERMUTATIONS TOUGHIES **Unit 53

Answers

1a) 5040 **b)** 840 **c)** 343

2a) 2,400,000 **b)** 5,760,000

3) 64

4) 120

5) 40,320

6) 10! arrangements x 10 seconds for each arrangement divided by 24 x 60 x 60 seconds in a day = 420 days (÷ 30 = 14months).

7a) 18278; (18279 would include a blank plate).

b) 475,228 ($26^4 + 26^3 + 26^3$)

8) 5^5

9) 24^{24}

10) n^n

11) n^4

12) n^n

13) n x (n – 1) x (n – 2) x (n – 3)

SOME PECULIAR THINGS ABOUT ZERO *Unit 54

1) $4 \times 0 =$ ____ 2) $0 \times 4 =$ ____ 3) $0 \times 0 =$ ____

4) $4 \div 0 =$ ____ 5) $0 \div 4 =$ ____ 6) $0 \div 0 =$ ___

These are simple exercises but they have two of them are frequently missed; exercise **4** and **6**.

Zero is like the present. It separates the past from the future but is a part of neither.

Zero is the answer to each of the other exercises, 1, 2, 3, 5.

7) If $8 \div 4 = 2$, then $2 \times 4 = 8$ and if $15 \div 5 = 3$, then ____________

8) Likewise, if $4 \div 0 = 0$, then ____________ and if $4 \div 0 = 4$, then _____________

Check all answers and see if you *disagree* with any.

9) Since $0 \div 0 =$ ___, then $0 =$ ___ $\times 0$

You might have been clever and answered 0 to exercise 9. It seems 0 is true if it works in both blanks. But try some other number in the blanks for exercise 9. Try 18.45. Does it work?

Try any number you want in exercise 9. Does it work? Of course. $\frac{0}{0}$ = ***any*** number.

$\frac{0}{0} = 0 \div 0$. If we think of the two little dots in $\div$ as the two numbers in any fraction, then it is easy to remember.

$\rightarrow$ 0/0 is the same as $\frac{0}{0}$, and the same as $0 \div 0$.

Because each of these symbols represent ***any*** number, each is called **i n d e t e r m i n a t e**. **Indeterminate** just means that a particular number **cannot be determined,** because any number works. Being indeterminate may be an important mathematical idea but a symbol for it cannot be accepted as a name for a number. $4 \div 0$, however, is simply impossible, because no number can be found to work.

Select values which will make each statement true:

10a) $0 \div$ ___ is indeterminate. b) $5 \div$ ___ is impossible.

c) $3 \div$ ___ is indeterminate d) $0 \div$ ____ is impossible

Some Peculiar Things About Zero *Unit 54

Answers 1 – 10

1) 0 **2)** 0 **3)** 0 **4)** Answered in text on page 1 **5)** 0

6) Answered in text on page 1 **7)** 3 x 5 = 15 **8)** then 0 x 0 = 4, 4 x 0 = 4

9) any number, same number again

10a) 0 **b)** 0 **c)** Impossible to make this indeterminate

d) Impossible to make this impossible! Is it possible to make it possible? _____

In the following examples, letters are used in the place of numbers which you must discover to make the statement true.

Illustration (1) $\frac{1}{a-3}$ is impossible. Answer: a = 3

The expression is impossible when a = 3 since $\frac{1}{3-3}$ = 1/0 (which is impossible).

Illustration (2) $\frac{a-2}{b-5}$ is indeterminate.

Answer: a = 2 and b = 5 (since we then have 0/0 which is indeterminate?)

11) $\frac{1}{c-4}$ is impossible. c = ______

12) $\frac{1}{2.8-m}$ is impossible. m = ________

13) $\frac{d-3\frac{1}{2}}{5}$ = 0. d = ___________

14) $\frac{c-5}{d-6}$ is indeterminate. c = _______ d = _______

15) $\frac{0}{7-t}$ is indeterminate. t = _______

SOME PECULIAR THINGS ABOUT ZERO *Unit 54

Illustration (3) Make this expression $\frac{0}{5-c}$ = 0. Here, we want c to be any value *except* 5 so that the denominator will not be 0. **c ≠ 5**, meaning c does not equal 5.

16) $\frac{0}{c - 1.3}$ = 0 __________

17) $\frac{6}{2\frac{1}{2} - b}$ is possible __________

18) $\frac{a - 2}{b - 3}$ = 0 ____________________

19) $\frac{a - 2}{b - 3}$ is impossible ____________________

20) $\frac{a - 2}{b - 3}$ is indeterminate ____________________

True or False:

21) $\frac{1}{5/8 - .625}$ is impossible ____

22) $\frac{0}{5/8 - .625}$ is indeterminate ____

23) $\frac{33\ 1/3\% - 1/3}{62\ 1/2\% - 3/8}$ is indeterminate ___

24) $\frac{3/25 - .09}{11}$ = 0 ___

25) $\frac{28\% \text{ of } 16^{10}}{11}$ is impossible ____

26) b/0 is impossible if b ≠ 0 ____

If any of the following are possible, give the answer. If impossible or indeterminate, so state.

27) $\frac{1/2 - 1/3}{0}$ ______________

28) $\frac{22\% \text{ of } 0}{80\% \text{ of } 0}$ ______________

29) $\frac{5 \times 0}{3}$ ___________

30) $\frac{87\ 1/2\% \text{ of } 7/8}{50}$ ____________

31) $\frac{.94 - .085}{5\%}$ ______________

32) $\frac{a-b}{a+b}$ when a = b _____________

____

33) $\frac{10}{a - b}$ when a = b ____________

34) $\frac{6.2 - 2\ 1/2}{37 \times 0}$ _______________

SOME PECULIAR THINGS ABOUT ZERO *Unit 54

Answers 11 – 34

The answer to the fourth line on page 2 is "yes" (any $n \neq$ **0)**

11) $c = 4$ **12)** $m = 2.8$ **13)** $d = 3\frac{1}{2}$ **14)** $c = 5$ and $d = 6$ **15)** $t = 7$

16) $c \neq 1.3$ **17)** $b \neq 2\frac{1}{2}$ **18)** $a = 2$ and $b \neq 3$

19) $a \neq 2$ and $b = 3$ **20)** $a = 2$ and $b = 3$

21) T **22)** T **23)** F **24)** F **25)** F **26)** T

27) Impossible **28)** Indeterminate **29)** 0 **30)** 49/3200

31) 17.1 **32)** 0 **33)** Impossible **34)** Impossible

COMBINATIONS **Unit 55

Suppose we want to know how many different groups we can get from a group of five objects. Let's take the five girls Joyce, Melody, Elizabeth, Brianna and Rebecca. We will call them J, M, E, B and R. One combination of three girls we could have is J, M and E. This group is the same combination as J, E and M, but not the same permutation. We know that this group of 3 girls has 3x2x1 = 6 permutations.

1) Of course, every single other group of 3 girls also has ____ permutations. That is 6 times as many permutations as combinations.

Tell the number of different combinations of the five letters taken three at a time.

Illustration 1: In a group of five girls, how many combinations of three girls are possible?

Solution:

$$\frac{5 \times 4 \times 3}{3 \times 2 \times 1} \quad \begin{array}{l}\text{(Number of arrangements possible)}\\ \text{(Number of arrangements of three objects)}\end{array}$$

2) $$\frac{5 \times \overset{2}{\cancel{4}} \times \overset{1}{\cancel{3}}}{\underset{1}{\cancel{3}} \times \underset{1}{\cancel{2}} \times 1} = \_\_\_\_ \quad \text{combinations}$$

Illustration 2: From a group of eight boys, how many different basketball teams (of five players) might be made? We can see here that the arrangement of the boys on the team does not matter.

Solution: $$\frac{8 \times 7 \times 6 \times 5 \times 4}{5 \times 4 \times 3 \times 2 \times 1} \quad \begin{array}{l}\text{Number of arrangements of 8 things taken 5 at a time}\\ \text{Number of ways in which 5 things can be arranged}\end{array}$$

$$\frac{8 \times 7 \times \overset{3}{\cancel{6}} \times \overset{1}{\cancel{5}} \times \overset{1}{\cancel{4}}}{\underset{1}{\cancel{5}} \times \underset{1}{\cancel{4}} \times \underset{1}{\cancel{3}} \times \underset{1}{\cancel{2}} \times 1} = 56 \text{ different teams}$$

You might have noticed in the in the two illustrations that there were as many factors in the numerator as the denominator. We will look at other examples to see if this idea is a general one.

COMBINATIONS **Unit 55

Illustration 3: How many different groups of three people can be made from a committee of four?

Solution: $\dfrac{4 \times \cancel{3}^{1} \times \cancel{2}^{1}}{\cancel{3}_{1} \times \cancel{2}_{1} \times 1}$ $\dfrac{\text{Number of permutations}}{\text{Number of permutations of 3 things}}$ = 4 groups of 3

A little thought will convince you that the number of factors in the denominator will always be the same as the number of factors in the numerator.

Because numerators and denominators will have the same number of factors in certain problems about combinations, we can make the work easier by writing the denominators in <u>reverse order.</u>

Illustration 4: How many different baseball teams (of nine players) could be formed from a group of 12 people? Solution: First write the numerator—the number of permutations of 12 things taken 9 at a time: <u>12 x 11 x 10 x 9 x 8 x 7 x 6 x 5 x 4</u>

Then, you write the denominator which has the same number of factors as the numerator, beginning at the left. Put 1 directly under the 12 and continue with the x 2 x 3, etc. You can then cancel (or cross reduce) to your heart's content. You might even find it easier to write the denominator <u>first and backwards</u>. If the idea helps, use it. The answer is 1320.

4) How many different basketball teams of 5 players might be made up from a group of 7 girls? ______

5) How many different hands of 3 cards each can be dealt from a pack of 52 cards? __________

6) How many groups of 6 letters can be made using the letters in the word "chairmen"? ______

COMBINATIONS **Unit 55

7) In a class having 15 members, how many different groups of five could be chosen to sit in the front row? _______

8) In a class of 20 girls, how many different pairs of girls might be formed? _____

9) A bag contains 8 balls, each of a different color. How many different pairs could be withdrawn?_____

10) In a club of 15 members, how many different committees of 3 members each could be chosen? _____

11) How many different chords of 3 notes each is it possible to strike on an 88 key piano?

____

12) Twenty-five people are at a party, not counting the host, and all are strangers to each other. The host wants to be sure that each person meets each other person at the party. How many introductions must she make? _____ (Note: The problem is much like number 8.)

COMBINATIONS

**Unit 55

Answers

1) 6 **2)** 10 combinations

3) 20 **4)** 21 **5)** 22,100 **6)** 28 **7)** 3003

8) 190 **9)** 28 **10)** 455 **11)** 109,736 **12)** 300

COMBINATIONS TOUGHIES ***Unit 56

Review Summary

The number of possible combinations of three objects from a group of five is:

$$\frac{5\times4\times3}{1\times2\times3}\quad\frac{\left\{\begin{array}{l}\text{Number of permutations of}\\ \text{5 objects taken 3 at a time}\end{array}\right.}{\text{Number of permutations of 3 objects}}$$

1) In a group of eight different signal flags, how many different signals could be made using 3 flags at a time? _____

*2) If eight lines are draw on the chalkboard, no two lines are parallel, no three lines intersect in one point, and each line intersects each other line before running off the board, in how many points will these lines intersect? (Hint: Each pair of lines has a point of intersection.) ______

**Illustration:* In a certain math class there are 12 boys and 8 girls. How many groups could be made up containing 5 boys and 2 girls? Each group must differ from each other group by at least one person.

Solution: It is necessary to break the problem into two parts.

First, we find out how many groups of five boys we can have.

$$\frac{\overset{1}{\cancel{12}}\times11\times\overset{1}{\cancel{10}}\times9\times8}{1\times\underset{1}{\cancel{2}}\times\underset{1}{\cancel{3}}\times\underset{1}{\cancel{4}}\times\cancel{5}} = 792 \text{ groups of 5 boys}$$

Second, we find out how many groups of two girls we can have: $\dfrac{\overset{4}{\cancel{8}}\times7}{1\times\underset{1}{\cancel{2}}}$ = 28 groups of 2 girls

Therefore, we can have 28 x 792, or 22,176 different *groups* of 5 boys and 2 girls.

*3) How many different groups containing three boys and four girls can be obtained from a class of twelve boys and six girls? _______

Find how many groups of 3 boys can be made and then how many groups of 4 girls.

COMBINATIONS TOUGHIES ***Unit 56

*4) A box contains 10 objects, all of different size, shape and color. Four are made of plastic and six of wood. How many different groups containing two wood and two plastic could be formed? ____

→ The symbol ${}_7P_5$ indicates the number of permutations of 7 objects taken 5 at a time.

→ The symbol ${}_7C_5$ indicates the possible number of combinations of 5 objects taken from a group of 7.

5) What symbol would be used to indicate the possible number of combinations of:

a. 3 objects from a group of 9? ______

b. n objects from a group of t? ______

6) ${}_nP_t$ is used to indicate the number or permutations of n objects taken t at a time.

a) What symbol would indicate the number of permutations of 6 objects taken in pairs? ___

b) Evaluate: ${}_6P_4$ =____ c) Evaluate ${}_6C_4$ = ____

d) Using this symbolism, what symbol would indicate all the possible permutations of 5 objects? _____

e) Evaluate ${}_4P_4$ _____ f) Evaluate ${}_6P_3$ _____

g) Evaluate ${}_6C_3$ _____

7) Circle each true statement:

a) ${}_qP_q = q!$ b) ${}_tC_q = t!/q!$ c) ${}_tP_q = t!/q!$ d) ${}_tC_t = 1$

8) If there are eight notes in the musical scale:

a) How many different melodies could be could be composed using each note only once? (This is a permutation problem.) ______________

b) Within the scale, how many chords could be struck using 4 notes at a time?_______

9) <u>Yes or no?</u>

In evaluating ${}_nC_p$ when n and p are given as natural numbers, is it possible to select values for n and p such that the answer is <u>not</u> a whole number? ____

10) The Wildcat Club has 6 members. In order to adopt a rule, the club must have a 2/3 majority vote. In how many ways might this be obtained? ______

COMBINATIONS TOUGHIES ***Unit 56

11) Using the four primary colors red, yellow, green and blue, what is the number of colors which may be formed:

a) by picking 3 colors at a time? ____

b) by picking *any* number of colors from 1 to 4? This means adding 4 results.____

*12) A woman stands in the doorway of a living room where a party is being held. She observes each guest shake hands with each other guest. If she observes a total of 120 handshakes, how many guests were at the party? _____ (Trial and error O.K.)

Answers

1) 56 **2)** 28 **3)** $\frac{6\times5\times4\times3}{4\times3\times2\times1}\times\frac{12\times11\times10}{3\times2\times1} = 3300.$ **4)** 90

5a) ${}_9C_3$ **b)** ${}_tC_n$

6a) ${}_6P_2$ **b)** 6 x 5 x 4 x 3 = 360 **c)** 15 **d)** ${}_5P_5$ **e)** 24 **f)** 120

g) 20

7) a and d should be circled. **8a)** 40,320 **b)** 70

9) No **10)** 15 **11a)** 4 **b)** 15 **12)** 16

MODULAR ARITHMETIC 1 *Unit 57

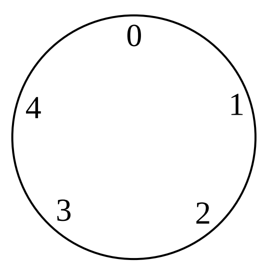
$$4 + 2 \equiv 1$$

What kind of arithmetic is this?

Included in the branch of mathematics called the Theory of Numbers, mathematicians sometimes find it useful to use arithmetics which are different from the arithmetic that we normally use. The ideas behind their arithmetics may not be advanced; they are in fact often quite simple.

The easiest way to see some of what modular arithmetic is about is to think of a clock or watch which has hands. If the time is 11:00 A.M. and we add three hours, the resulting time is 2:00 P.M. This can be expressed simply as $11 + 3 \equiv$ **2,** mod 12. Usually mathematicians read this "11 plus 3 is congruent to 2, modulo 12." The meaning of that word "congruent" is similar to "equals on a clock". We will have another definition later. Modulo is usually abbreviated as "mod". In this case the *modulus* is 12. That is, every time we get around to 12, we start over. This is the way modular arithmetic works.

We can easily make up an arithmetic with a different modulus, say 5. It is represented by the "clock" on the right. Notice that we put 0 at the top instead of 5.



In arithmetic mod 5, do these additions:

1) $4 + 2 \equiv$ ___ "Four plus two is congruent to ___ mod 5"

2) $3 + 4 \equiv$ ___ 3) $2 + 0 \equiv$ ___

4) $1 + 3 \equiv$ ___ 5) $2 + 4 \equiv$ ___

MODULAR ARITHMETIC 1 | *Unit 57

6) Using the table below, complete the addition table, mod 5. Notice that 4 + 1 ≡ **0.**

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2					
3					
4					

7) Make an addition table mod 7. Remember that the numbers in mod 5 table were 0 - 4.

+	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

Notice the many striking patterns in the completed table.

Give these answers mod 7:

8a) 6 + 3 ≡ ___ b) 5 + 5 ≡ ___ c) 6 + 4 ≡ ___ d) 3 + 2 ≡ ____ e) 4 + 3 ≡ ___

f) 3 + 6 + 5 + 3 ≡ ___ g) 6 + 6 + 6 ≡ ___ *h) 6 + 6 + 6 + 6 + 6 + 6 + 6 ≡ ___

Calculate:

9) 8 + 4 mod 9 ≡ ____ 10) 4 + 2 mod 5 ≡ ____ 11) 3 + 1 mod 6 ≡ ___

12) 1 + 1 mod 2 ≡ ___ 13) 5 + 6 mod 8 ≡ ___ ***14)** 8 + 8 + 9 + 9 + 8 + 8 mod 10 ≡ ___

***15)** 6 + 2 + 2 + 5 mod 10 ≡ ___ 16) 145 mod 12 ≡ ___

MODULAR ARITHMETIC 1 *Unit 57

Answers 1 – 16

1) 1, 1 **2)** 2 **3)** 2 **4)** 4 **5)** 1

6)

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	**2**	**3**	**4**	**0**	**1**
3	**3**	**4**	**0**	**1**	**2**
4	**4**	**0**	**1**	**2**	**3**

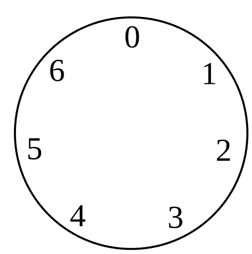
7) Addition **Modulo 7**

+	0	1	2	3	4	5	6
0	**0**	**1**	**2**	**3**	**4**	**5**	**6**
1	**1**	**2**	**3**	**4**	**5**	**6**	**0**
2	**2**	**3**	**4**	**5**	**6**	**0**	**1**
3	**3**	**4**	**5**	**6**	**0**	**1**	**2**
4	**4**	**5**	**6**	**0**	**1**	**2**	**3**
5	**5**	**6**	**0**	**1**	**2**	**3**	**4**
6	**6**	**0**	**1**	**2**	**3**	**4**	**5**

Once again, notice the patterns horizontal, vertical and diagonal.

8a) 2 **b)** 3 **c)** 3 **d)** 5 **e)** 0 **f)** 3 **g)** 4 **h)** 0

9) 3 **10)** 1 **11)** 4 **12)** 0 **13)** 3 **14)** 0 **15)** 5 **16)** 1



To subtract, we can use the table or go backwards on the clock. Thus, in **mod 7** arithmetic, since $4 + 5 \equiv 2$, then $2 - 5 \equiv$ **4,** and $2 - 4 \equiv$ **5**

17) Also, since $3 + 4 \equiv 0$, then $0 - 4 \equiv 3$, and 0 - ___ ≡ ___

Another way to look at the subtraction idea is to go backwards on the clock; that is, in mod 12, 2 o'clock minus 5 hours = 9 o'clock, which is $2 - 5 \equiv 9$ mod 12. We see then, that it is possible to subtract larger numbers from smaller numbers without using negative numbers when we are working in modular arithmetic. Do these **mod 7:**

18) a) $1 - 5 \equiv$ ___ b) $5 - 0 \equiv$ ___ c) $1 - 2 \equiv$ ___ d) $2 - 4 \equiv$ ___ e) $6 - 4 \equiv$ ___ f) $3 - 6 \equiv$ ___

MODULAR ARITHMETIC 1 *Unit 57

Here are some examples already worked out in modular arithmetic. Discover the modulus.

19) $7 + 3 \equiv 1$ mod ___ 20) $2 + 6 \equiv 0$ mod ___ 21) $8 + 9 \equiv 4$ mod ___

22) $3 + 3 \equiv 2$ mod ___ 23) $11 + 10 \equiv 8$ mod ___

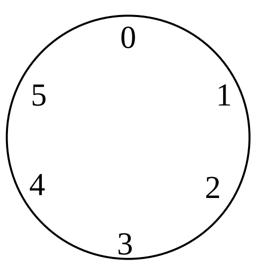
Answers 17 – 23

17) 3, 4 **18a**) 3 **b)** 5 **c)** 6 **d)** 5 **e)** 2 **f)** 4

19) 9 **20)** 8 **21)** 13 **22)** 4 **23)** 13

Multiplication is the next idea in modular arithmetic. In case you are beginning to think that you are back in the early grades learning about addition and multiplication, you can get some satisfaction by knowing that modular arithmetic is used in cryptography to encode and decode secret messages, to make discoveries about very large numbers, and to shed light on some mathematical ideas called groups and fields. In fact, many web sites seem to reflect the desire among writers to get very theoretical early in their writing about modular arithmetic. You can have a taste of all this in the unit Congruence and Modular Arithmetic and its references.

For now, to get a look at multiplication we will consider the modulus 6.



It is easy to calculate 3 x 5 mod 6. Clockwise starting from 0, count up to 5 three times, each time starting your new count from the end of the previous count. Your answer should be 3. It is interesting in this case how the count slips back one unit each time because it is 3 x **5**. Sometimes it is easy to get started wrong, especially at the beginning of any count after the first.

24) 2 x 4 mod 6 ≡ ___ 25) 3 x 4 mod 6 ≡ ___ 26) 4 x 5 mod 6 ≡ ___ 27) 5 x 3 mod 7 ≡ ___

MODULAR ARITHMETIC 1 *Unit 57

This method works fine for small numbers but gets impractical quickly with larger numbers. You might have already discovered something like the following:

8 x 4 = 32 (Not modular).   8 x 4 $\equiv$ 2 mod 6 (Modular).

32 requires 5 spins of 6 fully around the mod 6 dial, plus 2. This means that 32 $\equiv$ 2 mod 6.
This is read "32 is congruent to 2 mod 6", as we said on page 1.
Also, note that 32 $\div$ 6 = 5 remainder 2.

Read this at least one more time.

Answers 24 – 27:

24) 2 **25)** 0 **26)** 2 **27)** 1

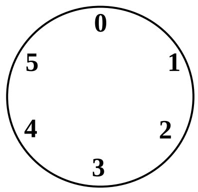
28a) Fill table below, multiplication modulo 6.

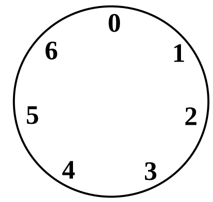
x	0	1	2	3	4	5
0	0					
1						
2						
3			0			
4						
5						

b) This multiplication mod 7 is the final table you will be asked to complete.

x	0	1	2	3	4	5	6
0	0						
1							
2							
3			6				
4							
5							
6							

MODULAR ARITHMETIC 1 *Unit 57

Mod 6 

Mod 7 

29) 6 x 4 ≡ ___ Mod 7.

You can tell from your answer in the table on the previous page that the answer is 3. On page 2 you had some repeated additions which needed repeated counting around a clock, a technique you could use for modular multiplication. Or, you could say 6 x 4 = 24 which is 3 more than the next lower multiple of 7 (21). This tells you that 6 x 4 ≡ 3 mod 7. Check it on the clock above. Likewise, 3 x 20 mod 7 is 4 more than a multiple of 7.

30) 3 x 20 ≡ ___ mod 7 31) 3 x 4 ≡ ___ mod 7 32) 3 x 5 ≡ ___ mod 7

33) 9 x 4 ≡ ___ mod 7 34) 9 x 3 ≡ ___ mod 7 35) 6 x 9 ≡ ___ mod 7

36) 8 x 8 ≡ ___ mod 10 37) Use table backwards to do 5 ÷ 3 ≡ ___ mod **7**

38) 10 x 10 ≡ ___ mod 12 39) 6 x 6 ≡ ___ mod 5 40) 1 ÷ 4 ≡ ___ mod 7

Don't spend too long on this one:→ 41) 1 ÷ 4 ≡ ___ mod 6 42) 3 x 13 ≡ ___ mod 37

43) 3 x 2 ≡ ___ mod 6 44) ______ = 7 mod 10 ← Find 3 numbers between 1and 100.

45) Note that multiplication mod 6 has zeros in the body of the table but x mod 7 does not.

Why not? ____

MODULAR ARITHMETIC 1 *Unit 57

Answers 28 – 45

28a) Mod 6

x	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

b) Mod 7

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

29) 3 **30)** 4 **31)** 5 **32)** 1 **33)** 1 **34)** 6 **35)** 5

36) 4 **37)** 4 **38)** 4 **39)** 1 **40)** 2 **41)** There is no answer to 1 ÷ 4 Mod 6.

42) 2 **43)** 0 **44)** 7, 17, 27 or any three numbers between 0 and 100 ending in 7.

45) 7 is a prime modulus; it has no factors besides 1 and itself that would multiply to give sevens scattered throughout the body of the table.

The unit following this one provides some interesting extensions of the ideas you have seen here.

MODULAR ARITHMETIC 2 **Unit 58

Reaching an understanding of this topic is an excellent mental exercise. It often happens that we think we know a thing, but feel unsure when we meet it again in slightly different surroundings. Be patient in regaining focus.

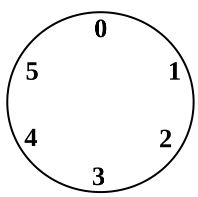
Multiplication Mod 6

x	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

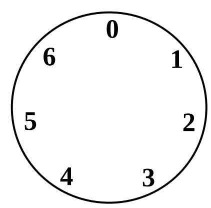
Multiplication Mod 7

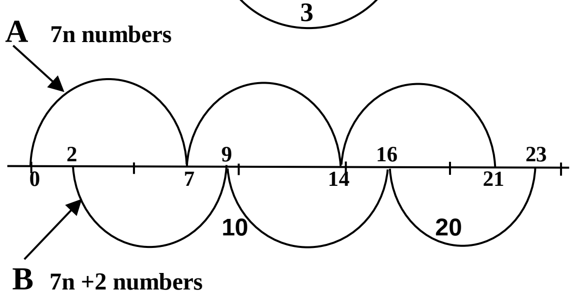
x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

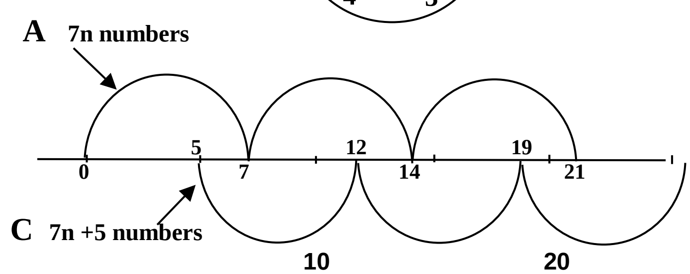
Mod 6



Mod 7







1) In each of **A, B** and **C,** the *difference* beween any two consecutive landing places is ___.

2) If you divide *any two* numbers in set **A** by 7, the remainders will be ___;

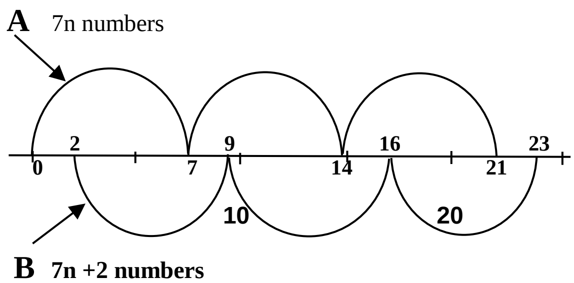
in set **B** the remainder will be___; in set **C** the remainder will be ___.

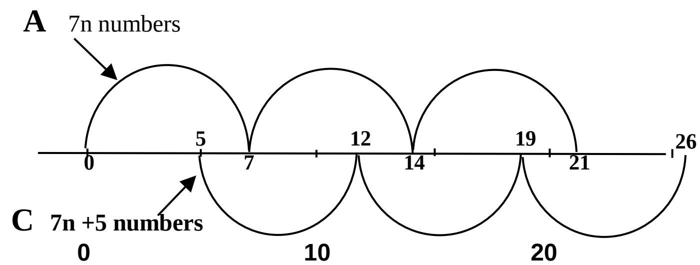
3) What is the remainder if a 7n + 2 number is divided by 7? ___;

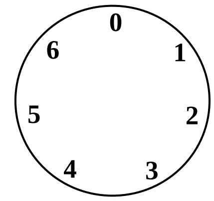
if a 7n + 5 number is divided by 7? ___; if a 7n is divided by 7? ___.

4) 700 is easily recognized as a multiple of 7, so 700 mod 7 ≡ ___

MODULAR ARITHMETIC 2 **Unit 58







mod 7

5) T/F: 714 $\equiv$ 721 mod 7 ___

6) T/F: 714 and 721 have the same start and stop points on the mod 7 clock. ___.

7) T/F: 7,035 and 14,049 have the same start and stop points on the mod 7 clock. ___.

8) T/F: 7,035 $\equiv$ 14,049 on the mod 7 clock. ___. Note: If any two positive integers are divisible (evenly) by 7, they are said to be *congruent with respect to modulus 7.*

9) 7,036 and 14,050 have the same start and stop points on the mod 7 clock, even though they are not multiples of 7. T/F: 7,036 $\equiv$ 14,050 mod 7___.

10a) The truth of exercise 9 (and it *is* true) suggests that if the *difference* of two numbers is divisible by 7, they *might* be multiples of 7, but *are* congruent with respect to the modulus 7, T/F: ___.

b) What are/is the start and stop point(s) for 16 and 30 on the mod 7 clock? ________

11) T/F: If two positive integers, say 16 and 30, have a difference divisible by 7, they also will have the *same remainder when divided by 7*, and so are congruent mod 7 ___

12) T/F: It is easier to see that the numbers 39,540,324 and 39,540,366 are congruent mod 7 by subtracting them mentally instead of dividing each by 7 to find the remainders ___.

13) T/F: 2,821,072 and 3,514,079 each have a remainder 2 when divided by 7. Here, this is an easier test for congruence than subtracting and seeing if the difference is divisible by 7. ___

MODULAR ARITHMETIC 2 **Unit 58

Answers 1 – 13

1) 7 **2)** 0, 2, 5 **3)** 2, 5, 0 **4)** 0

Exercise 5 to 13 are all True, except **10b)** is 2.

Do not consider negative numbers.

---

14) T/F: $1317 \equiv 1301 \bmod 8$. ___

15) Name 3 numbers for which 1317 and 1307 are congruent. ___________

16) T/F: $a \equiv b \bmod 2$ if a and b are both even: ___; a and b are both odd: ___; a is even and b is odd: ___; a is odd and b is even: ___.

17) **Example:** Give all positive integers such that each is congruent to 5, mod 3 (Try only non-negative integers.) Answer: 8, 11, 14, 17, . . .

Follow the example for each of the exercises below

a) ⬡ $\equiv 2 \bmod 2$ ____________________ b) ⬡ $\equiv 3 \bmod 5$ ______________________

c) ⬡ $\equiv 11 \bmod 17$ _________________ *d) ⬡ $\equiv 2^2 \bmod 2^3$ ______________________

Answers 14 – 17

14) T **15)** 10, 5, 2 **16)** T, T, F, F

17a) 0, 2, 4, 6 …. **b)** 3, 8, 13, 18 …. **c)** 11, 28, 45, 62 …. **d)** 4, 12, 20, 28 ….

Recap:

1. Two integers are congruent mod **M** iff their difference is divisible by **M** and, .
2. Two integers are congruent mod **M** iff they have the same remainder when divided by **M**

Note: As always, divisible means with remainder 0, and iff means if and only if. Also: While 0 is divisible *by* any other number, dividing by 0 is not possible. So, a modulus cannot be zero.

MODULAR ARITHMETIC 2 **Unit 58

*18) Example: ◯ + 2 ≡ 5 mod 3 (Do not try negative numbers).

Answer: 6, since 8 – 5 is divisible by 3.

But there are more than just the 6; there are also 9, 12, 15, . . .

*a) ◯ + 3 ≡ 11 mod 7 ___,___,___,___, . . .

*b) 4 + ◯ ≡ 12 mod 4 ___,___,___,___, . . .

*c) 7 + ◯ ≡ 13 mod 11 ___,___,___,___, . . .

*19) In the following, the two integers given are congruent with respect to some modulus that is left for you to discover. Give every possible modulus in each case (ignore 1).

a) 10 ≡ 2 mod _________ b) 14 ≡ 0 mod _______ c) 36 ≡ 17 mod ________

d) 13 ≡ 7 mod _________ e) 86 ≡ 10 mod _______________

*20) T/F: If the difference between T and A is divisible by K, then:

a) K ≡ A mod T ___

b) T ≡ K mod A ___

c) A ≡ T mod K ___

Note: Call answers to 21) and 22) true only if always true.

21) T/F: If p ≡ q mod m, then:

a) p – m is divisible by q. ___

b) $\dfrac{p-m}{q}$ is an integer. ___

**22) Find some number N, such that when the number is squared, the result is congruent to 13 mod 17. That is, find N where $N^2 \equiv 13$ mod 17. Give the two positive answers. _______. Remember that N is asked for, not N^2.

Carl Friedrich Gauss became very interested in problems like exercise 22. There are many internet sites under the name “Modular Arithmetic”. It is a popular topic to write about but most articles quickly get theoretical. There is a calculator in one site which will compute modular examples even if they have extremely large numbers. <Wolfram/alpha/computational knowledge engine>. If you want to use exponents as in 2^{400}, type “2” and then “^”.

MODULAR ARITHMETIC 2 **Unit 58

You might also find that <Modular arithmetic/Introduction/aopswiki> is something you can follow if you give just a quick look at tedious parts of long explanations. Understanding the distributive property of multiplication over addition will help but is not absolutely necessary:

a) 56 + 154 = **7 x 8 + 7 x 22 = 7 x (8 + 22)** = 7 x 30 = 210

b) 7 x 86 = **7 x (80 + 6) = 560 + 42** = 602

Answers 18 – 22

18a) 8, 15, 22, 29 **b)** 0, 4, 8, 12 **c)** 6, 17, 28, 39

19a) 8, 4, 2 **b)** 14, 7, 2 **c)** 19,

d) 6, 3, 2 **e)** 76, 38, 19, 4, 2

20a) F **b)** F **c)** T

21a) F **b)** F

22) 8, 9

CARD FLIPS, GROUPS AND FIELDS Unit 59

A	B	**I**
D	C	

D	C	**H**
A	B	

B	A	**V**
C	D	

C	D	**R**
B	A	

1. Get a 3" by 5" card or cut one out of thin cardboard. Exact dimensions are not important. Label the card clockwise in the corners as shown in the top diagram. This is arrangement **I.** Memorize this one but not the others.
2. Arrangement **H is** formed from **I** by flipping the card about the ***H****orizontal* dotted line so you are looking at the back side. Letter the back side as shown in **H.** Of course the card is now upside down and backside forward with respect to **I.** Be sure that the A in **H is** directly behind the A on the opposite side, and the same with the other three letters. When in the **H** arrangement, each of the letters on your card is upside down to its partner on the opposite side.
3. Return the card to the **I** arrangement.
   The **V** arrangement is formed from **I** by flipping the card end to end about the ***V****ertical* fine line. The letters are upside down to you, but each is right side up to its partner on the opposite side.
4. Return the card to the **I** arrangement.
   The **R** (for Rotate) arrangement does not use the back side. From I, simply spin the card 180 degrees and it will be upside down, with letters also upside down, but in the *letter positions* on the card shown in **R.**

1) Return the card to the **I** arrangement. Now move the card from **I** to the **V** arrangement. From **V**, move it to the **H** arrangement. The result is the same as the single move from **I** to ___. This important answer is **R**. We will symbolize the procedure this way: **V** o **H** = **R**. This says: "**V** followed by **H** is the same as **R"**
=

2) Try this one: **V** o **R** = ___. The answer is **H**.

V o **I** = **V.** Why? **I** is like 0 in addition. 0 makes no change. It allows the number to keep its identity, just as **I** provides *no move* in card flips. **V** o **I** = **V** says: "Making no move from the **V** arrangement, equals **V".**

CARD FLIPS, GROUPS AND FIELDS

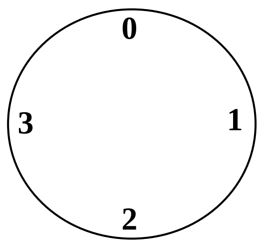
Unit 59

3) **I** o **V** says “No move followed by **V**”. **I** o **V** = ___. Thus, **I** o **V** = $\overbrace{\mathbf{V} = \mathbf{V}}^{\text{This is from exercise 2}}$ o **I.**

4) Finish the table having the operation “followed by”, O, and the elements **I, V, H, R.** Use your card to find answers that you have not worked out in the text. Be sure to notice patterns that develop in the table to see how parts of the table might continue. Then check your answers on the next page.

O	I	V	H	R
I			H	
V				
H		R		
R				

+	0	1	2	3
0				
1				0
2		3		
3				



We use 0 instead of 4 on this “4 clock”. If you can understand that 3+2=1, you are all set.

5) On a watch or clock having hands, 9 + 2 = 11, but 11 + 2 = 1. Also, 10 + 5 = 3.

a) 9 + 5 = ___. b) 9 + 8 = ___.

6) Now fill in the addition table Mod 4. (That is, do it using the 4 clock.)

7) On the face of “4 clock”, above right: a) 3 + 2 = ___. b) 3 + 3 = ___ *c) 3+3+3 = ___

8) Now finish the addition modulus 4 table, above. Again, watch patterns develop. Not only are there patterns in each of the two tables, but there are also patterns which are shared, or are at least similar.

a) Heading and first row are identical in each table.

b) Left side headings and first column are identical in each table.

c) There is an **I** in each column and in each row in the Flip table and a **0** in column and each row in the mod 4 table.

d) T/F: (V + H) +R = V + (H + R) in the Flip table ____

e) T/F: (1 + 2) + 3 = 1 + (2 + 3) in the mod 4 table ____

CARD FLIPS, GROUPS AND FIELDS Unit 59

Answers 4 – 8

o	**I**	**V**	**H**	**R**
I	I	V	H	R
V	V	I	R	H
H	H	R	I	V
R	R	H	V	I

5a) 2 **b)** 5

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

7a) 1 **b)** 2 **c)** 1

8) Answers with Discussion.

a) H o R = V and 2 + 3 = 1 are examples from each table and this can be done for ***any pair*** of elements in *each* table. This is called ***closure.*** Each set of elements is closed for its operation.

b) Heading and first row are identical in each table. **This** shows that there is an ***identity element*** **I** in the card flipping table and an ***identity element*** **0** in the mod 4 table.

I o (a move) = that move, and 0 + n = n, that same element

Left side and first column are identical in each table. Same as part a, but "backwards". It works the same way.

(a move) o **I** = that move, and n + 0 = n.

CARD FLIPS, GROUPS AND FIELDS — Unit 59

c) There is an **I** in each column and in each row in the Flip table and there is a **0** in each column and each row in the mod 4 table. Recall that in the integers -5 + 5 = 0. If two elements add up to the identity 0, the numbers are *opposites* of each other. The presence and location of **I** and **0** show that every element in each table has an opposite.

d) T **e)** T

If you inspect these two statements in d) and e), you will see that, given three elements with the appropriate operation, it does not matter how they are ***associated***, or grouped. This ***associative property*** is true for both the Flip table and the Mod 4 addition table.

So we have four properties that are true of any group:

i) Closure. A set of elements, say a,b,c, etc., and an operation, say $\oplus$, such that a $\oplus$ d = m. (You do not have to go outside the set to get the answer to *any* example.)

ii) Identity element. There is an element I such that a $\oplus$ I = a.

iii) Inverse. For any element "a", there is an opposite, say "b", such that a $\oplus$ b = I.

iv) Associative. for any a, b, c, (a $\oplus$ b) $\oplus$ c = a $\oplus$ (b $\oplus$ c).

v) It is often true, but not necessary, that a $\oplus$ b = b $\oplus$ a. If true, the operation is called c***ommutative,*** and the group is called "abelian".

You might wonder why a systematic listing of properties is made of such differing sets as numbers in addition modulo 4, and flips about center lines in a 3 x 5 card. This is exactly what one aspect of mathematics is about: seeing the common characteristics which relate objects and operations into systems in mathematics.

9) Tell whether the each of the following sets and operations is a group:

a) Y/N (Yes/No): The positive integers under addition ___.

The word "under" is used to designate the operation.

b) Y/N: The integers under addition ___.

c) Y/N: The integers under subtraction ___.

d) Y/N: The positive real numbers under multiplication ___.

e) Y/N: The real numbers under division ___.

f) Y/N: The numbers mod 4 under addition ___.

CARD FLIPS, GROUPS AND FIELDS Unit 59

Answer 9

9a) No. No inverse exists for any positive integer among themselves.

b) Yes. **c)** No. Subtraction is not associative $(5-3)-1 \neq 5-(3-1)$

d) Yes **e)** No. Division is not associative. **f)** Yes.

Also basic to mathematics is the topic "Fields". It takes two related groups to make a field. Examples of two such groups are:

1. The real numbers under *addition*, and
2. The real numbers under *multiplication* but without 0.

These two groups are related in a very specific way which is probably not new to you:

For any real numbers a, b, c, $a \times (b + c) = a \times b + a \times c$.

This is called the **distributive property** of multiplication over addition and you have been using it in multiplication.

$$\begin{array}{r} 32 \\ \underline{\times 4} \\ 128 \end{array} = 4 \times (30 + 2) = (4 \times 30) + (4 \times 2)$$

$$= 120 + 8$$

$$= 128$$

multiplication distributes over addition

As we said of modular arithmetic in unit 58 page 4, writers seem to enjoy picking Groups as a topic. <Wikipedia/Theory of Groups> will, in a few glances, demonstrate the depth and importance of Groups in mathematics. You will find much that is familiar in another web site, <Dog school of mathematics/Introduction to Group Theory>. This will be helpful if you are interested in extending your knowledge in this large area.

PRIMES 1 | **Unit 60

A prime number is a natural number that has no factors (divisors with remainder 0) other than itself and one. Examples are 5, 17, 31, 41 (not 51), etc.. Some numbers "look" prime but are not**:** 9, 39, 51, 93. The four numbers just mentioned have a common factor greater than one.

1) What is that common factor?____

2) Also, 91 = ____ x ____ .

3) Complete the chart to the right, ending with the number 102. Next, go through the whole chart **circling** all the prime numbers, remembering that 1 is not a prime (1 factor) but 2 is (2 factors)**.** There is one clear pattern which will appear as you proceed, but its predictive value is very limited. Think about this as you make and correct the completed chart.

Correct to here (on next page)

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72

4) The numbers 11, 17 and 23 are each 5 more than a multiple of 6. In earlier units we identified such numbers as 6n + 5 numbers. We could also call them 6n – 1 numbers. 13, 19 and 31 could be called 6n + 1 numbers. List all the 6n + 1 primes between 40 and 100. There are 6 of them.

____

5) The reason there are no primes in the third column is that all members are multiples of ____.

6) Starting with the second **row**, no matter how far the chart is extended, will a prime ever be found in any column other than the first and fifth? _______

7) Therefore, all primes are either 6n + 1 numbers or ____ + ____ numbers.

8) All 6n numbers are in column number ____.

9) A natural number which is not prime is *composite* (kom-paw-zit). It has factors other than itself and 1. List all the factors of 20, including 1 & 20. There are six of them.

{1,20, 2, ____________}

PRIMES 1 **Unit 60

10a) If a composite number has <u>many</u> factors, as does 72, a system is helpful so that all factors are listed. Write the factors in pairs (a system we already know) with the first pair being smallest/largest, then next-smallest/next-largest, as in {1,72, 2,36, 3, . . etc.}. There are 12 factors, 6 pairs:____

Notice how the pair members get closer to each other as they get bigger.

b) Similarly, write the factor pairs for 60: {__________________________________

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	**80**	**81**	**82**	**83**	**84**
85	**86**	**87**	**88**	**89**	**90**
91	**92**	**93**	**94**	**95**	**96**
97	**98**	**99**	**100**	**101**	**102**

11) In both parts of ex. 10, the pairs reach a point where no more searching for factors is needed (unless you missed an early one). *That pair was 6, 10 for 60 and was ____, ____ for 72.

12) What is the closest together pair of factors for

a) 48 _______? b) 30 _______?

c) 36 _______? d) *71 ___________

13) It often comes as a surprise that you do not have to check for factors all the way to <u>half</u> of a number to find its factors. Half of 80 is ____ but you need only to check to the pair _______ to find all its factors. Note that the closest pair is the square root of the number as in 36, or close to it as in 72.

14) In the chart's top row, the numbers 4 and 6 are composite and 2, 3, and 5 are prime. The only even prime is _____**.** 1 is neither prime nor composite**;** its only factor is ____.

15) By grouping the natural numbers in 6 columns you can see that certain patterns show up. But, is there anything predictive in these columns regarding where primes will show up? Not much. Except for 2 and 3, all primes are 6n + 1 numbers or 6n + ____ numbers. Too bad not **all** the numbers in those columns are primes. Especially with large numbers, it is not conclusive evidence that a number is prime to show that it is in one of these two columns.

PRIMES 1 **Unit 60

Again, the only conclusive way to establish a number as prime is to show that it has no factors other than itself and one. All of column 2 numbers end with an even number, so they are all divisible by 2. But 2 itself is prime (because it has exactly 2 factors). Column 3 numbers are divisible by 3**,** but 3 itself is prime. Similar to column 2 numbers, all column 4 numbers are even, so they are not primes. Column 6 contains all numbers divisible by 6 (and of course by 2 and 3), so they are composite.

When testing whether a large number is prime, it is very helpful to use divisibility rules**:**

-- If the sum of the digits of a number is a multiple of 3, the number is also.

-- If the sum of the digits of a number is a multiple of 9, the number is also.

-- If the last digit of a number is even, so is the number.

-- If a number ends in 5 or 0 it is divisible by 5.

-- If a number ends in 0 the number is a multiple of 10 (and 2 & 5).

16) Between 100 and 110 there are exactly four prime numbers. It is extremely rare to find an interval so rich in primes. Use rules of divisibility to eliminate the composites in that interval and name the four primes: ________________________________.

17) In the interval between 110 and 120, use divisibility rules to eliminate composites and name the numbers remaining**:** ____ and ____. The few divisibility rules we use will not necessarily reveal composites with prime divisors. You will need to investigate the remaining numbers by dividing by primes. Final remaining prime(s) between 110 and 120**:** __________**.**

Mathematicians have been fascinated by prime numbers for centuries. Many have tried to find a test to tell whether a number is prime without doing a large number of divisions. Nobody has succeeded. Once the rules for divisibility have been exhausted, we cannot be sure, for example, whether 113 is prime without doing some dividing. Now we turn to a larger number. Is 1601 prime? It is not easy to tell. Certain divisors can be eliminated easily**:** 2, 3, 5 and their multiples. (27 is certainly not a factor because 3 is not.)

18) Which of the following numbers are clearly not divisors of 1601? Circle them by using rules for divisibility, but do not divide. Use a fact like neither 3 nor 5 is a factor, so 15 is not.

a) 15 b) 18 c) 21 d) 28 e) 29 f) 31 g) 35 h) 37 i) 39 j) 40 k) 400

PRIMES 1 **Unit 60

19) We still do not know whether 1601 is prime. What numbers must we test by division to be sure? Remember that we do not have to test 2, 3, or 5 or their multiples. List every number less than 40 that must be divided into 1601 in order to discover any prime factors of 1601.

____

You can use the chart (page 2 of this unit) to check your answers above because they consist of all primes from 6 to 40. There are 9 of them, starting with the number 7.

20a) We have greatly reduced the work needed to test whether a number is prime by using rules for divisibility and then actually dividing only by primes. How far must we test any sizeable number to see if it is prime? Many would insist that "halfway" is the point where the decision may be made. What do *you* say?

____.

20b) But we do better than that by merely testing to the square root of the number, rounding the answer if necessary. In earlier factoring by pairs we discovered the factor pair beyond which we did not have to test. This was always at, or near the square root of the number we were testing. If you test 1601 beyond 40 you will get answers smaller than 40 that earlier testing had already found by listing pairs. So, in testing for primes, you do not have to list pairs beyond the square root**;** you need only to realize the redundancy of going beyond the square root. (Look up "redundancy" if you don't remember the meaning. You have seen it before.) So, is 1601 prime? ______

21) Use the strategies you have learned – rules for divisibility by 2, 3, 5, 6, 9 and 10, and dividing by *primes* no farther than the square root of the number – determine whether each of the numbers below is prime. If the number is prime, say so. If not, state its smallest factor (not counting 1) and that factor's partner in the factor pair. Use a calculator. We should mention that there is a rule for 9 just like the rule for 3. If the sum of the digits of a number is 9 or a multiple of 9, the number is also divisible by 9. The rule for 3 would always catch this but you would have to apply it twice.

a) 397 __________ b) 40011 ____________ c) 899 __________ d) 913 ___________

e) 887 ________ f) 611 ______________ g) 437 __________ h) 1223 __________

PRIMES 1 **Unit 60

Answers 1 – 21

1) 3 **2)** 7 x 13 **3)** Answers on page 2. **4)** 43, 61, 67, 73, 79, 97

5) 3 **6)** No **7)** 6n + 5 (or 6n – 1) **8)** 6

9) {1,20 2,10 4,5}

10a) {1,72 2,36 3,24, 4,18 6,12 8,9} **b)** {1,60 2,30 3,20 4,15 5,12 6,10}

11) {8,9} **12a)** {6,8} **b)** {5,6} **c)** {6,6} **d)** Prime

13) 40 {8,10} **14)** 2 1 **15)** 5 **16)** 101, 103, 107, 109 **17)** 113, 119; 113

18a) 15 **b)** 18 **c)** 21 **d)** 28 **g)** 35 **i)** 39 **j)**40 **k)** 400

19) 7, 11, 13, 17, 19, 23, 29, 31, 37 **20a)** No, only to or near the square root of the number. b) Yes

21a) Prime **b)** 3, 13337 **c)** 29, 31 **d)** 11, 83

e) Prime **f)** 13, 47 **g)** 19, 23 **h)** Prime

<u>Study</u>**:**

The prime factorization of 45 = 9 x 5 = 3 x 3 x 5 = 3^2 x 5

Another way**:** 45 = 3 x 15 = 3 x 3 x 5 = 3^2 x 5

The prime factorization of 280 = 28 x 10 = 4 x 7 x 2 x 5 = 2^3 x 5 x 7

(There are other ways.) = (and rearranging to <u>ascending</u> order of prime factors)

Perhaps you use "factor trees" or some other system to find prime factorizations. This can be streamlined by factoring in <u>convenient</u> stages, as above. You might not proceed in the order shown. Other orders work well, too.

Another example: You finish it, using reasonable shortcuts.

360 = 36 x 10 = 6 x 6 x 5 x 2 = ______________________________________

= ______________________________________

(Use exponents, with prime factors in ascending order)

22) Give prime factorizations, ending with prime factors in order and with exponents.

Make good choices to begin, and continue that mental work as you progress:

a) 480 = ____

b) 900 = ____

c) 1240 = ____

d) 1230 =____

e) 2718 = ____

Note: you will need to test 151 to see whether it is prime. Test mentally and remember not to exceed its square root.

*23) Give prime factorizations:

a) 1800 = ____

b) 271,800 = ____

c) 960 = ____

d) 2480 = ____

In earlier work, you were sometimes asked to list all the factors of a number and as an aid in checking, you were told how many factors there were in the number. You are now in a position to learn how many factors a number has without listing and counting them. The answer to exercise 23b, above, is 2^3 x 3^2 x 5^0 x 151. From this we could discover some factors: 2^2 x 5 = 20, 2 x 3^2 x 5 = 90, but it would be quite long and easy to miss some. It is possible to tell from the prime factorization that 281,800 has 72 factors. To understand this, we need to know some ideas about permutations, or the number of different ways things can be arranged or events can happen.

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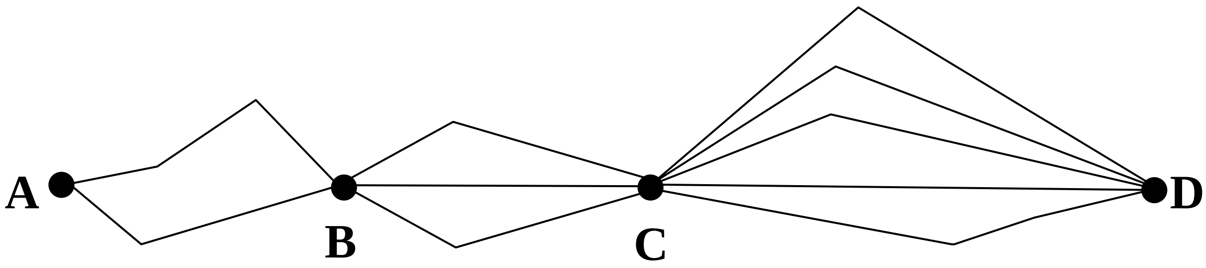


(This is similar to a diagram in Permutations 1, Unit 51)

There are two routes from **A** to **B**, three from **B** to **C** and five routes from **C** to **D.** There are 3 routes from B to C to continue the upper route from A to B and again 3 routes from B to C to continue the lower route A to B.

24a) How many **different trips** can be made from A to B? _____

b) Since there are three ways for continuing the trip from B to C, there are how many ways to make the part of the trip from A to C ? (Not 5) _____

c) The larger number of choices for the final leg C to D allows a total of ____ choices for planning the entire trip A to D**.**

Having arrived at C by one of the 6 possible routes, you can see that the 5 routes C to D offer the traveler 30 choices for planning the entire route A to D**.** $2 \times 3 \times 5 = 30$. In much the same way we can create particular factors of a number by making choices from its *prime* factorization $2^3 \times 3^2 \times 5^2 \times 151$ (= 271,800)**.**

The choices 2^2 and 5^1 (4×5) give the factor $2^2 \times 5$ (= 20).

Think of the "zeroth" power of 3 and of 151 as $3^0 = 1$ and $151^0 = 1$.

Mathematically, for any non-zero number **n**, $\mathbf{1 = n/n = n^1/n^1 = n^0}$, $\mathbf{n^0 = 1}$**.**

From above 271,800 we can choose the particular factor $2^3 \times 3^2 \times 5^0 \times 151^1 = 10{,}872$. This may be interesting but how does it contribute to the ease of finding that there are 72 factors in the original number 271,800 ? ($2^3 \times 3^2 \times 5^2 \times 151^1$)

PRIMES 1 **Unit 60

All the possible choices are summarized as follows:

2 can have an exponent of 0, 1, 2, or 3. (4 choices)

3 can have an exponent of 0, 1, or 2. (3 choices)

In exercise 25, you complete the summary for 271800. Don't forget the "extra" exponent choice of 0. The prime factorization was 2^3 x 3^2 x 5^2 x 151.

25a) 5 can have an exponent of ____,____ or ____ (____ choices)

b) 151 can have an exponent of ____, or____ (____choices)

c) The total number of choices possible for 271,800 is 4 x 3 x ___ x ___ = _____

More Answers: Your processes will probably differ from those shown, but there is a "Fundamental Theorem of Arithmetic" which says **"**A whole number may be factored into primes in only one way**".** Order does not matter but it is helpful and we ask for it**.**

Answers 22 – 25

From introduction, page 5**:** 360 = 36 x 10 = 6x6 x 5x2 = 2x3 x 2x3 x 5x2 = **2^3 x 3^2 x 5**

22a) 480 = 6x8x10 = 2^3x 2x3x2x5 = **2^5x3x5** **b)** 900 =20x45 = 2^2x5x3^2x5 = **2^2x 3^2x 5^2**

c) 1240= 2^3x5x31 **d)** 1230 = 2x3x5x 41 **e)** 2718 **= 2 x 3^2x 151**

23a) 1800 = 2^3x 3^2x 5^2 ***b)** 271,800 = 2^3 x 3^2 x 5^2x151

c) 960 =16x6x10=**2^6x3x5** **d)** 2480 = 2^4x 5x 31

24a) 2 **b)** 6 **c)** 30

25a) 0, 1, 2; (3 choices) **b)** 0 or 1(2 choices) **c) 3, 2; 72 (choices)**

26a) The prime factorization of 1350 = 2^{---} x 3^{---} x 5^{---} .

b) The number of factors of 1350 is ____ x ____ x ____ = _____

Note**:** The answer is **not** 1 x 3 x 2

27) Use exercise 22 answers above to help find the number of factors of each number:

a) 480 _____ b) 900 _____ c) 1240 _____ d) 1230 _____ e) 2718 ______

PRIMES 1 **Unit 60

28a) $17 \times 101^3 \times 151^2$ is a large number. How many factors? ______

b) $2 \times 3^3 \times 5^2$ is not so large. How many factors? _____ (Are you remembering to increase each exponent by 1, especially where the exponent does not appear and is understood to be 1?)

*c) $3 \times 4^3 \times 6^2$ is slightly larger. How many factors? _____ The correct answer is ____. Trick question! Do you see why (?). This factorization is not expressed in primes. Fix your answers as needed after completing the misleading non-prime factoring handed to you. That job is OK as far as it goes.

29) Sample: True or false: 75 is a factor of $2^2 \times 3 \times 5^4$. You are being asked to see if you can show that the factors of 75 will "fit", or be "contained in", $2^2 \times 3 \times 5^4$. So, since the factorization of 75 is 3×5^2, it will "fit" or divide into $2^2 \times 3 \times 5^4$. Answer: **True** ($3 \times 5^2 = 75$) Below, circle the letter of each which is a factor of $2^2 \times 3 \times 5^4$:

a) $2 \times 3 \times 5^2$ b) $2 \times 3^2 \times 5^4$ c) 2 d) 25 e) 100 f) 300 g) 11 h) 8

30) $39{,}151 = 7^2 \times 17 \times 47$. Does 39,151 have 12 factors? _____(It does if the above is a prime factorization. Is it? ______)

> ***For exercise 31*****:** When each of two numbers has a factor of 5, so does their sum. When one number has a factor of 5 but not the other, their sum does not. When neither number has a factor of 5, the sum might or might not. (As in 12 and 13)

31) Circle each of these (the letter) which has 7 as a factor. This question is the same as asking whether a seven jumper starting at 0 would land there. In other words, is the number (or the answer to operations of x, +, exponents, etc.) a multiple of 7?

a) 777 b) 778 c) 779 d) 784

e) 777 + 14 f) 777 + (19 x 7) g) 777 + 56 h) 707 + (6 + 7 + 5)

i) 777 + (a very large prime) j) 7707 x (a very large prime)

k) 7077 + (7 x (a very large prime))

PRIMES 1 **Unit 60

Answers 25 – 31

26a) $^{1\ \ 3\ \ 2}$ **b)** 2 4 3**;** 24

27a) 24 **b)** 27 **c)** 16 **d)** 16 **e)** 12

28a) 24 **b)** 24 ***c)** 3 x 4^3x 6^2=2^8x 3^3 ;9 x 4= 36

29) circle a, c, d, e, f, Note**:** b fails only because of 3^2.

30) yes, yes **31)** Circle all **except** b, c, h, and i.

An operation in mathematics that might surprise you is **!**.

3! is called 3 factorial and 3! = 6**,** 4! = 24, 5! = 120; 8! = 40,320; 12! = 479,001,600.

13! is read "13 factorial" means 13! = 13 x 12 x 11x 10 x 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1.

Use information you have one line above on this page for a shortcut to calculate 13!.

32) 13! = ______________

Unless you have at least a ten digit readout on your calculator you will run into scientific notation with 13!, certainly with 14!. You do not need your calculator to find 13!, but it is helpful if you can find the "!" key**.** The ! operation can be done on a computer's calculator, but you might need the "!" on your computer's keyboard. The times sign may be a raised dot on the computer's calculator. Sounds complex but it is not difficult.

In the following, no calculator.

33) Use sensible shortcuts: a) $\frac{4\times3\times2\times1}{3\times2\times1}$ = b) 8!/ 7! = c) $\frac{5!\times6!}{4!\times7!}$ =

d) 8!/ (4!x 4!) = e) $\frac{100!}{100\times99!}$ =

34) What is the number of factors in 2^5 x 3^2x 5 x 7 ? ___________________

*35) How many factors has 10! ? Recall that the prime factorization of 10! is needed, but there are shortcuts. Instead of writing 10 x 9 x 8 . . . etc., write 5 x 2 x 3^2 x 2^3 etc., (the prime factorization of each factor), gather them together, order them, and find the number of factors 10! has. (See exercise 26a and b for a reminder). _____________

36) Circle the letter of each expression divisible by 6: (That is even <u>and</u> divisible by 3).

a) 130,215,115 b) 23! c) 23! + 18 d) 2^3 x 3^3 x 5^2

e) 60! + 5 f) 60! + 5! g) (60 + 5)! h) 30 x 1783

Answers 32 – 36

32) 6,227,020,800 **(**13! = 13 x 12!), useful if you happen to have 12! handy)

33a) 4 **b)** 8 **c)** 5/7 **d)** 70 **e)** 1

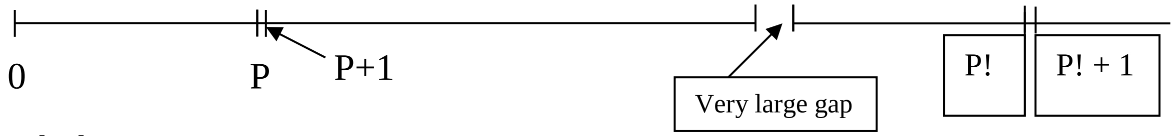
34) 72

35) 270 factors. This is a surprisingly large number of factors that can be derived from 10! - - over 3 ½ million factors.

36) Only a) and e) should <u>not</u> be circled

We have not been including 1 in our calculations because multiplying by 1 to *any* power would not make a difference.

PRIMES 2 ***Unit 61



No calculator

1) **T/F:** **(**Note: All factorials above 9! end in 2 or more zeros)

Reminder: Two consecutive natural numbers share no factors.

a) 19! has 17 as a factor. ______

b) 19! + 1 has 17 as a factor. _____

Correct a and b now (Answers are on next page)

c) 18! + 2 has 2 as a factor. _____

d) 19! + 2 has 2 as a factor. _____

e) 19! + 1 has (16 x 15) as a factor._____

**f) 19! + 2 has the factor (16 x 15). _____

Correct to here.

g) 1,000 is a factor of 1,000,001!. _____

h) 1,000 is a factor of 1,000,000!. _____

i) (1,000,001! + 1) has 737 as a factor. _____

j) 1,000,001! has 737 as a factor. _____

k) (999,999 x 999,998) is a factor of 1,000,001!. _____

l) If 100 is a factor of 200 and of 700 then it is a factor of 700. _____

m) If 13 is a factor of 26 and of 260, then it is a factor of their sum. ____

n) 2517 is a factor of (1,000,000! + 2517). _____

o) 2517 is a factor of (1,000,001! + 2517). _____

Correct g – o now.

p) 1000! has more than 2000 factors. _____

q) (8 x 21 x 37) is a factor of (100! + 1) _____ (Important idea)

r) 1,000! > (is greater than) 5,000,000,000. _____

s) If two numbers are 5 apart, no jumper greater than 5 can exist in that interval more than once._____

t) 18! and 17! share more than 35 common factors. _____

u) (2 x 3 x 5 x 7) + 1 has no prime factor less than 11._____

v) 1,000,001! is 1,000,001 times as big as 1,000,000! _____

Correct p – v now.

PRIMES 2 ***Unit 61

Answers 1 (with comments)

All of the statements for exercise 1 should have been answered T or F. None was NF. Finish your corrections thoughtfully, doing your best to understand why your incorrect answers, if any, are wrong before going on.

1a) True

b) False**.** You know that consecutive natural numbers do not have any common factors except 1**.**

c) True; each has 2 as a factor.

d) True**;** same as c.

e) False, (16 x 15) is a factor of 19!, so not of the number consecutive to it -- 19! + 1.

f) False: 19! ends in zeros**;** adding 2 would prevent 15 from being a factor.

g) True **h)** True**;** Note that 1,000,001! and 1,000,000! are not consecutive numbers.

i) False because (j) is true. **j)** True **k)** True **l)** T **m)** T

n) True; each has the factor 2517. **o)** True; same as n.

p) True. Most of the 1000 numbers multiplied to get 1000! have several factors each.

q) False. (8 x 21 x 37) is a factor of 100!, therefore not of 100! + 1.

r) True. Just the largest four multipliers to get 1000! are enough to exceed 5 billion.

s) True **t)** True. They share all the factors of 17!

u) True**;** (2 x 3 x 5 x 7) shares no factors with (2 x 3 x 5 x 7) + 1**;** they are consecutive.

v) True.

17! is a large number. It is larger than 355 trillion. If you have access to a computer you can use its calculator to verify the exact value as 355,687,428,096,000. You can do this by finding the ! key on the calculator and entering 17! **.** See if that produces the same 355,687,428,096,000. “!” is on the computer keyboard, perhaps not on the calculator.

If you do not have an adequate computer calculator, then accept 355,687,428,096,000 as the correct value for 17!. Knowing that value is not as important (don’t memorize it) as remembering

PRIMES 2 ***Unit 61

how we got it**:** 17! = 17 x 16 x 15 x . . . 3 x 2 x 1. The 17! expression above, or its reverse, is the best to work from.

Here it is**:** 17 x 16 x 15 x 14 x 13 x 12 x 11 x 10 x 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1

2) Write the prime factors of 17! in descending order without exponents. Write each only once.

17, 13,______________________________.

*3a) Now that you have discovered all the primes in 17!, use exponents to tell how many times each prime appears, forming the prime factorization (Reminder: The prime factorization of 600 is 2^3 x 3 x 5^2.) __________________________.

**b) Use part a) to find the number of factors in 355,687,428, 096,000. Recall that you were told earlier that 17! had more than 10,000 factors. Also, 2^{15} can occur 16 different ways including 2^0. ______________________________

4) **T/F:** or NF (Not Fair, even with a calculator).

a) 13 is a common factor of 17! and 13._____

b) 23 is a common factor of 17! and 23._____

c) Aside from 1, 31 is the only common factor of 32! and 31._____

d) Aside from 1, 12 is the only common factor of 17! and 12._____

*e) Aside from 1, 3 is the only common factor of 23! and 6. _____

f) 329 is a factor of (1001! + 329)._____

g) 330 is a factor of (1001! + 330)._____

h) 331 is a factor of (1001! + 331)._____

i) 1461 is a factor of (1001! + 1461)._____

j) 1001 is a factor of (1001! + 1001)._____

k) 1001! ends with quite a few zeros._____

*l) 92 is a common factor of 100! and 60. ___

m) 94 is a common factor of 95! and 93. _____

n) 194 is a common factor of 200! and 195!___

PRIMES 2 ***Unit 61

Answers 2 – 4

2) 2, 3, 5, 7, 11, 13, 17

3a) $2^{15} \times 3^6 \times 5^3 \times 7^2 \times 11 \times 13 \times 17$ **b)** 10,752 factors

4a) T

b) False. It would be true if 23 were a factor of 17! But we can see in exercise 2 that 23 was **not** among the prime factors of 17!**.** Why not?? 17! is a **large** number and you might ask w*hy* its ***prime*** factors cut off so quickly at 17. (See 2 and 3a answers above.) More soon.

c) True **d)** False. All factors of 12: 1,2,3,4,6,12, are also factors of 17!

e) False. Others are 6, and 2 **f)** True **g)** True **h)** True

i) False, but not really fair because we don't know whether 1461 is a factor of 1001! (It is not.)

j) True **k)** True **l**) False **m**) False **n**) True

It is not a coincidence that the answers to 2 and 3a both end with 17 in some form. 17 is the largest prime in 355,687,428,096,000 because that number was built by multiplying only the numbers 1 to 17. But the prime factorization with its $2^{15} \times 3^6$ etc, shows how such a small group of numbers with some of them raised to various powers, can be put together to form a number like 355 trillion. If primes like 13 and 17 are given large exponents and multiplied together, they could produce very large numbers but the largest **prime factor** would still be 17. Multiplication of natural numbers, prime or not, **cannot produce primes different from those being multiplied.** **(**Obvious but easily overlooked.)

5a) What is the largest prime factor of 51! ? (Tricky question, but not difficult) ______

b) What is the largest perfect square factor of 51! ? _____

As numbers get larger and larger, the occurrence of primes becomes more and more rare**.** But this does not happen in a regular way. The author's 1970+ Kay Pro computer gave the following results**:** 300,821 is a prime and the very next odd number, 300,823, is also a prime ("Twin primes")**.** After that there are no primes in the next 27 numbers.

PRIMES 2 ***Unit 61

6a) An unusual group of numbers is the interval from 300,490 to 300,500. There are four primes in that short stretch of eleven numbers. Name the four prime numbers (Merely eliminate the composites with divisibility rules.) _______________________________.

In the 101 numbers from 10,000,000 to 10,000,100 inclusive, there are only two primes. That is not many compared to the 25 primes between 0 and 100. But then there is a surprise from 900,000,000 to 900,000,100 inclusive. There are five primes in that interval of 100 numbers. Even so, the trend is definite. The "farther out" we go, the fewer primes we find. Perhaps we should expect this because large numbers look "more divisible" than smaller numbers.

Answers 5 – 6

5a) 47 **b)** 49

6) 300,491; 300,493; 300,497; 300,499

Because the distance between primes seems to get larger and larger, we might ask the question: "*Somewhere,* are there 1000 consecutive numbers having no primes at all?" (or 1,000,000 consecutive numbers, etc.?). This question can be answered. In example 4f – 4h, we identified a few consecutive non-primes:

$$1001! + 329, \quad 1001! + 330 \quad \text{and } 1001! + 331.$$

The list on the next page contains those consecutive composites and asks you to insert four more.

PRIMES 2 ***Unit 61

Fill in the consecutive composite numbers (non-primes) before and after those given**.** The **arrows** point at each answer position.

7)

→ a)______________________

→ b)______________________

1001! + 329

1001! + 330

→ c)______________________

|

|

(Many left out)

|

→ d)______________________

1001! + 1000

____

Each number is an expressed sum.

It is important to understand truly that the numbers listed are indeed consecutive and that, within the expressed sum, there really *is* a factor other than itself or one. Because 329 is a factor of 1001! and of 329, then it is a factor of the sum of 1001! and 329**.** Likewise, *each* of the two members of the sum 1001! + 330 is divisible by 330, so the sum is composite, etc**.** This is the Distributive Property.

We could go on listing many such consecutive numbers but how far can we go with confidence? Since all numbers up to 1001 appear as multipliers (factors) in 1001!, we can continue the list without interruption up to 1001! + 1001**.** All of these numbers will be *composite*.

The *largest* common factor of these is 1001! + 1001.

The answers to exercise 7 are on the next page. Check them now and return.

Using exercise 7, we now have consecutive composites from 1001! + 327 to 1001! + 1001 (if you have corrected your work for 7a – 7d). Note that if we go any farther to 1001! + 1002, we lose the pattern of 1001! being coupled with a number easily known to share a factor with it. In our quest for 1000 consecutive composites we can back up and begin with 1001! + 2.

We will not go back to 1001! + 1 because testing it for being prime is not a simple matter. That is, any number divided by one gives itself as the answer. Doing so would not distinguish whether the number is prime or non-prime.

So, we will start with a friendlier, but still appropriate, number**:**

8a) 1001! + 2 is definitely divisible by ____.

We start here because we ruled out 1001! + 1.

PRIMES 2 ***Unit 61

b) 1001! + 3 is definitely divisible by ____.

*c) So, we have a continuous list of definite composite numbers from 1001! + 2, to 1001! + 1001.

How many consecutive numbers is that? _______

Answer 7

7a) 1001! + 327 **b)** 1001! + 328 **c)** 1001! + 331 **d)** 1001! + 999

You might think "Shouldn't we have endings from 2 to 100**2** instead of 2 to 100**1** to get 1000 numbers when we subtract? Nope. If you hold in your hand the 1 – 10 of hearts, how many cards do you hold? "Ten" you say? Correct. But wait! If we subtract 1 from 10 we get 9, not 10. If we subtract when we really intend to count, we should remember to add 1 to the subtraction answer. So it is here (and in ex. 8c, answer below).. If we subtract 2 from 1001 we get 999**;** add 1 and you will have **1000** numbers and avoid a very common error. Welcome to the club if it was your error too.

9) We have found 1000 consecutive numbers which contain no primes!! They are the composite numbers from 1001! + 2 to 1001! + 1001**.** It was quite a search, especially because **each** of these numbers is so unimaginably large. Use the calculator in your computer (Go to Accessories list) to find out how many zeroes following the number 4 are needed to express 1001! ____________.

**10) Use patterns we used on these pages (to the extent that you need them) to name an interval of numbers containing:

a) 1,000,000 consecutive composite numbers_____________________________

b) 5,000 consecutive non -primes____________________________________

PRIMES 2 ***Unit 61

Answers 8 – 10

8a) 2 **b)** 3 ***c)** 1000 . If this is nor clear, see above answer to exercise 7 starting with "If you hold in your hand the 1 – 10 of hearts, . . ." ***9)** 2,570

****10a)** 1,000,001! + 2 to 1000,001! + 1,000,001 **b)** 5001! + 2 to 5001! + 5001

There seems to be no doubt that <u>any</u> particular interval of non-primes could be designated, no matter how large. This raises the question of whether or not the primes "thin out" to the point of disappearing. Do primes continue indefinitely or is there some limit beyond which there are no more? Your present level of skill and understanding is sufficient (if you have reached this point successfully) to participate in proving an answer to the question.

In the third century B.C., Euclid (You-klid), the man who organized geometry into a well-structured deductive body of mathematics, answered the infinity question about primes. He used the negative, or indirect method of proof, a kind not often used in his work on geometry, but which was central in Cantor's work on transfinite numbers.

Euclid assumed that there was indeed a last prime and then investigated the consequences of the assumption. We used this kind of proof in the unit on Cantor. For convenience, call the "last" prime P. It should be understood to represent a large number but without factors, aside from 1 and itself.

But first, do these refresher problems. Include NF for "not fair" when appropriate

11) List all the **prime** factors of these**:** 1 is not a prime.

a) 5____ b) 6_____ c) 7 ____ d) 10 _____ e) 25 ____ *f) 5! ____ *g) 6! _____

*h) 7! __________ *i) 10! ___________ *j) 25! ________________________

12a) List the prime factors of 100 _______________________

b) List the prime factors of 1,000,000,000,000. _______________________

*c) Give the largest prime factor of 118_____ ← Hint: First think of the smallest.

**d) Give the largest prime factor of 118! ______

*13) Describe the set of prime factors of 150! between 150 and 180.________________

*14) Name all factors of (100! + 1) which are smaller than 100_________________

*15) In exercise 13, why is 151 not a prime factor of 150!___________________________

PRIMES 2 ***Unit 61

Answers 11 – 15

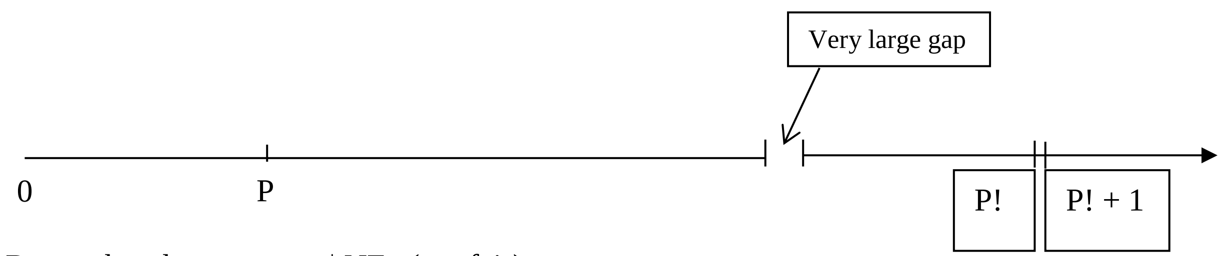
11a) 5 **b)** 2, 3 **c)** 7 **d)** 2, 5 **e)** 5 ***f)** 2, 3, 5

***g)** 2, 3, 5 **h)** 2, 3, 5, 7 **i)** 2,3,5,7 **j)** 2,3,5,7,11,13,17,19, 23

12a) 2, 5 **b)** 2, 5 **c)** 59 **d)** All numbers less than 118 are factors of 118! and the largest prime of these is **113**

***13)** None, or empty set ***14)** None, or empty set

***15)** All prime factors of n! are equal to or less than n. See page 4 under "text resumed" for more detailed explanation.



Remember the occasional NF. (not fair)

16a) List the common factors of 13! and 17 ________________________

b) List the common factors of 1300! and 24 ____________________________

c) List the common factors of 1300! and (1300! + 1). __________________

d) List the common factors of 13 and 17! ___________________

17) Suppose P is a large prime number (We don't know which number).

a) Is P! prime? ____ b) Is P! + 1 prime? _____ c) Is (P + 1)! prime? _____

d) Does P! end in zero? _______ e) Do P! and P! + 1 share any factors apart from 1?_____

Since P is a large prime, then P! is very large, ends in 0, and so is not prime. But P! + 1 is a very large number which ends in 1. (Do you see why?) There is no easy way we can know whether it is prime or not. Question 17b is NF, but "Can't tell" would also be good.

PRIMES 2 ***Unit 61

18) Suppose again that P is a large prime:** (Think patiently.)

a) Aside from 1, can P! + 1 have a factor smaller than P? _______

b) If P! + 1 is not itself prime, then it has at least one prime factor larger than _________.

c) What is the significance of the answer to part b? _______________________________

____

Answers 16 – 18

16a) None **b)** 24, 12, 8, 6, 4, 3, 2 **c)** None (consecutive numbers) **d)** 13

17a) No **b)** Can't tell **c)** No **d)** Yes **e)** No, they are consecutive.

****18a)** No. P!, being *consecutive* to P! +1, has barred its own factors from use by P! + 1

b) P. **c)** But P has been assumed as the *last* prime so that assumption is false.

This is the proof that there is no last prime!

To recap, P! + 1 is much larger than P and, if P! + 1 is not itself prime, it must have at least one prime factor that is *larger* than P (See example 18). Therefore, the assumption that there is a last prime P, leads to the conclusion that a still larger prime exists. Therefore, there is no last prime!!!

19) **A prime chase.** You might like better a different road to the conclusion that there is no last prime: Fill in all the blanks.

Row A	2 x 3	x 5	x 7	x 11	x 13	x _____	etc.
Row B	6	30	210	______	________		
Row C	+1	+1	____	____	____		
Row D	7	___	____	______			

Correct all of your entries thoughtfully. Answers on the next page.

(For a real challenge, try seeing what the chart tells us *before* doing exercises a – e.)

a) Row A lists consecutive ______________. Satisfy yourself that you understand how Rows B,C and D were obtained, though you might not see *why* it was done. The product 2 x 3 x 5 x 7 is given in row B under the x 7. It is 210.

b) The 211 in that column, being consecutive to 210, shares no factors with it. T/F ___

PRIMES 2 ***Unit 61

*c) So, 211 has no factors other than 1 and 211. T/F ____. d) Therefore, 211 is ________.

*d) Did the same things happen in the previous column (x 5) ?_____; in your (x 11) column? ____. No matter how far we extend the columns, is it a *certainty* that Row D will contain a prime number larger than the prime at the top of its column?_____

e) What has been proved regarding a last prime? ________________________________ ____.

This argument was adapted from Calvin Clawson, pages 147-148, Book List #6 and Constance Reid, pages 32-33, Book List #16.

Answer 19

Chart Answers:

Row A	2 x 3	x 5	x 7	x 11	x 13	etc.
Row B	6	30	210	2310	30030	
Row C	+1	+1	+1	+ 1	+1	
Row D	7	31	211	2311	(30031)	

Not asked for

19a) primes **b)** T **c)** T, prime **d)** Yes; yes, yes

e) There is no last prime. Any new prime (in order) in row A will produce a new and larger prime in row D.

FOUR SPECIAL PROBLEMS ***Unit 62

<u>For group or class (answers in Unit 63)</u>

1) Problem 1

<u>Materials Needed:</u> Three cards of one color and two of another, and pins or clips to attach a card to the back of a person's clothing. We will refer to the cards as three orange and two white.

Select three people from among volunteers to stand in front of the class facing away from each other. Pin a card on the back of each person. The class should watch; no volunteer should know what color is on her/his own back.

Attach a white card on the back of two volunteers' clothing and an orange card on the back of the third. Now the class knows the color for each volunteer but the volunteer does not.

The instructions, to be heard by all students, are told to the volunteers. *"Each should look at the backs of the other two but not at your own. (No signaling, either vocal or visual, may be given.) When you are sure that you know the color of your own card, return to your seat immediately."*

a) Of course, the student wearing the orange card will see exactly two white cards and sit down almost immediately. The other two will hesitate, but one of them (or both) will soon conclude that --- well, what does the second student conclude about her/his own card color, and why?____

____

b) Will the third student sit down very soon? Tell about his/her thoughts.

____

*b) The second situation is to attach one white card and two orange and give exactly the same instructions as in part a). Use different students. Tell what will happen and why.____

____

____

***c) The third situation is to attach orange cards to all three students. Tell what will happen.

____

____

____

FOUR SPECIAL PROBLEMS ***Unit 62

**2) Second problem: It has been said that the whole numbers are rooted in sets, demonstrated by the following:

$$\underset{0}{\{\ \}},\quad \underset{1}{\{\{\ \}\}},\quad \underset{2}{\left\{\{\ \}\{\{\ \}\}\right\}},\quad \underset{3}{\left\{\{\ \}\ \{\{\ \}\}\ \left\{\{\ \}\{\{\ \}\}\right\}\right\}}$$

Your task is to provide the "Sets" expression for 4.

$4 \longrightarrow$

**3) Third problem: From 12:00 noon until 12 midnight, how many times do the hands of a clock form a right angle with each other? _________

**4) Fourth problem: Two mathematicians were strolling down Mathechusetts Avenue looking for the address #1729 Mathechusetts Ave. One said to the other "The number 1729 is not very interesting."

The other replied, "Oh, but it *is* interesting. 1729 is the sum of two cubes in exactly two ways." This was enough to quiet the other mathematician until, a few minutes later, she came up with the two ways _______ + _______, and _______ + _______.

FOUR SPECIAL PROBLEMS (ANSWERS) Unit 63

1a) Seeing the only two white cards on the backs of the other two students, the orange card person will sit right down knowing the she/he must have the only remaining color, orange. The second to sit down realizes more quickly than the third that the first was so quick because he/she had seen two white cards.

b) (One white card, two orange). Each orange card student, seeing one of three orange and one of two white, hesitates. Each could think: "If mine was white, the student with orange color would see two white and sit down. Since she/he is not sitting down, mine must be orange", and sits. The other student with an orange card will sit down upon realizing why the first student sat.

c) (3 orange) All three students will hesitate, perhaps for quite a while. Finally, one student will sit, having seen that nobody else is sitting because we do not have situation a or b, that is, nobody has a white, so he/she has an orange card.

2) $\left\{ \{\}, \{\{\}\}, \left\{\{\}\{\{\}\}\right\}, \left\{\{\}\ \{\{\}\}\ \left\{\{\}\{\{\}\}\right\}\right\} \right\}$ **4**

3) 22 times. Now that you know the answer, maybe you can figure out how to come to that knowledge. Here is a hint (Maybe). Magellan's trip around the world had two (at least 2) notable events.

a) Magellan did not complete the trip because he died of a fever.

b) His crew, upon completing the trip to the home port, found that their own calendar disagreed with the calendar at home by one day. This fact is related to the answer of 22 for the hands of the clock: The minute hand is chasing the hour hand and *passes it* only 11 times in twelve hours, making right angles twice between passes. If Magellan traveled eastward in circumnavigating the globe he would have seen an "extra" sunset; westward would mean chasing the sun and be more like the clock problem, seeing one less sunset.

4) $1729 = 1728 + 1$. Note that the volume of a 1 foot cube is $12 x 12 x 12 = 1728$ cu. inches.

$1729 = 1000 + 729 = 10^3 + 9^3$

BOOK LIST Appendix I

The books listed here contain some ideas which are interesting, adaptable to the understanding of people who have had little or no algebra, and have importance to mathematics today or historically.

In most of these books, much of the content is well beyond Middle School or Junior High School level. Books marked with # are most likely to have *some* content which will draw the student's interest now and in the future. ## indicates high recommendation. Some of these books are out of print, but finding them is yet another service the Internet provides for the patient searcher.

1) ### Aczel, Amir D. *The Mystery of the Aleph.* Pocket Books. 1230 Avenue of the Americas, New York, NY, 10020 - The most readable book about Cantor himself and his work. Highly recommended.

2) Asimov, Isaac. *Asimov on Numbers.* Garden City, NY: Double Day and Co.,1977

3) # Barrow, John D. *Pi in the Sky*. Little Brown and Co.: Boston, 1994

4) Beckman, Peter, *A History of Pi*. St. Martin's Press: New York, 1971

5) # Bell, Eric Temple. *Men of Mathematics.* Simon and Schuster, 1937

6) ### Clawson, Calvin. *Mathematical Mysteries.* Plenum Press, 233 Spring Street, New York, NY 100137578, 1996. Rare gift for saying much in a few words. Very comprehensive, entertaining and informative.

7) # Hoffman, Paul. *The Man Who Loved Only Mathematics.* Hyperion: New York. The mathematician who spent his adult life traveling all over the world visiting his friends, almost all of them professional mathematicians. Eccentric and gifted**;** many poignant and comical situations are described.

8) # Hogben, Lancelot. *Mathematics for the Million*. W.W. Norton and Co.: New York, 1951

9) Ifrah, Georges, *From One to Zero. Harrisburg,VA:* R..R.. Donnelley and Sons, 1985

10) ## James and James, Editors. *Mathematics Dictionary. Princeton, NJ:* D. Van Nostrand Co. 1949

11) *# #*Jingh, Jaqjit. *Great Ideas of Modern Mathematics: Their Nature and Use;* Dover Publications, Inc. New York , 1959.

12) Kline, Morris. *Mathematics in Western Culture.* Oxford University Press. NY, 1953.

Book List — Appendix I

13) ## Newman, James R. *The World of Mathematics*, *Vol. I - IV*. Simon and Schuster: New York, 1956

14) ## Ogilvy, C. and Anderson, J. *Excursions in Number Theory*, Dover Publications, Inc.

15) # Pedoe, Dan. *The Gentle Art of Mathematics.* The MacMillan Company, New York

16) ## Reid, Constance. *A Long Way from Euclid.* Thomas Crowell Co. New York. 1963 - Unusually clear and enjoyable flow of ideas from Ms. Reid

17) ## Rucker, Rudy. *Infinity and the Mind.* Bantam Books. New York, 1982 - The infinity "bible". Not easy, but rewarding reading from a brilliant mind.

18) Stewart, Jan. *Nature's Numbers.* Basic Books, NY 10022-5299, 1995

19) Stwertka, Albert. *Recent Revolutions in Mathematics.* Franklin Watts. New York

Internet Sites are referenced within or at the end of appropriate units. Wikipedia was the single most often used site.

SUGGESTED TOPICS Appendix 2

Suggested List of Grade Eight Topics Not Fully Covered in the Units

If these units are being used as a class wide text, the listed additional topics probably warrant attention. See also the final three entries in the “Book List”.

- Specific strategies for solving "word" problems
- Bar, circle and line graphs and data
- Unit pricing
- Precision, relative error, accuracy
- Metric system and conversions
- Operations and conversions with time and measure units
- Algebraic equations and inequalities normally given to a class of this level
- Four fundamental operations with fractions, decimals and percents
- Probability
- Parallels, transversal, corresponding, alternate interior angles, complementary and supplementary angles
- Compass and straightedge constructions
- Similar triangles and proportions, indirect measure
- International dateline, time zones
- Symmetry
- Range of very elementary algebra skills
- Commutative, associative and distributive properties
- Perimeters, areas and volumes; especially circles, polygons, cones, cylinders and spheres
- Non-metric geometry**:** rays, half lines, planes, etc.
- Celsius, Fahrenheit, Kelvin scales

QUIZ: NUMBER GIANTS — Units 1 - 2

1) 5 cube = ____

2) 20 square = ______

3) 15 x 10^{20} = 15 followed by ___________________________

This is a 3

4) $(4^2)^3$ = _______

5) $(4)^{2^3}$ = _______

6) 283.0172 x 10^2 = _______________

7) 123.123 x 10^5 = ____________

8) 1,300,000 = ____ x 10^{---} (No decimals)

9) $\left(\frac{2}{5}\right)^3 =$

10a) $\left(\frac{4}{5}\right)^{603} \times \left(\frac{5}{4}\right)^{602} =$

***b)** $\left(\frac{5}{13}\right)^{9} \times \left(\frac{13}{5}\right)^{\boxed{}} = 2\frac{3}{5}$

11a) $\frac{5\times10^2}{(5\times10)^2} =$ ____

b) $\frac{10^{10}}{10^8}$ x 4.2 = _____

c) $\frac{5\times10^{32}}{10^{41}}$ x $\frac{10^{43}}{5\times10^{32}}$ = _____

d) $\frac{3\times10^{50}}{10^{20}} \times \frac{10^{21}}{6\times10^{50}} =$ _____

12) $(4/5)^3$ x $(5/2)^3$ = _______

****13)** $\left[\frac{2}{?}\right]^3$ x $\left[\frac{6}{4}\right]^2 = \frac{2}{3}$; ? = ____

QUIZ: BING, A HUMAN BINARY COUNTER AND MAGIC NUMBER CHART

Units 3 - 4

1) If four people stand and face the class in a row, the person at the ___________ (right, left) end, as the class sees them, responds to the "Bing" command. (This person will be called the <u>first</u> person.)

2) The signal for any *other* person is from the person (from the classes perspective) on his/her ______(left, right) who is putting his arm ________ (up, down).

3) We will designate a person as a box: $\boxed{}$. A person with one arm up will be $\boxed{1}$. $\boxed{0}$ will designate a person with no arm up. How many Bings will it take to give a line of people

a) like this? $\boxed{0}\boxed{1}\boxed{0}$ ____, and b) this? $\boxed{1}\boxed{0}\boxed{0}$ ____

4) In Unit 03 you dealt with up to 6 Bing positions: $\boxed{0}\boxed{0}\boxed{0}\boxed{1}\boxed{0}\boxed{0}$ Notice that this one shows the same number of Bings as the one just above it. How many Bings would it take to produce the set of positions shown below? $\boxed{0}\boxed{0}\boxed{0}\boxed{1}\boxed{1}\boxed{1}$ ____

5) How many for this? > > > > > > > $\boxed{0}\boxed{0}\boxed{1}\boxed{0}\boxed{0}\boxed{0}$ _____

6) Continue from exercise 5, starting with the successor to exercise 5.

$\boxed{}\boxed{}\boxed{}\boxed{}$ ____ $\boxed{}\boxed{}\boxed{}\boxed{}$ ____ $\boxed{}\boxed{}\boxed{}\boxed{}$ ____ $\boxed{}\boxed{}\boxed{}\boxed{}$ ____

7) Here is the strip from exercise 5: $\boxed{}\boxed{}\boxed{1}\boxed{0}\boxed{0}\boxed{0}$ It took 8 bings. This could also be said that 1000_2 ("One zero zero zero base 2") = 8. Also, $1000_2 = 8$. So, $1010_2 =$ ____

8a) $1111_2 =$ ___ **b)** $1011_2 =$ ____ **c)** $10001_2 =$ ____ **d)** $18 =$ __________ $_2$

9) $2^3 = 8$. **a)** $2^4 =$ ___ **b)** $2^6 =$ ____ **c)** $2^{---} = 256$ **d)** $2^{----} = 1024$ ***e)** $2^0 =$ ____

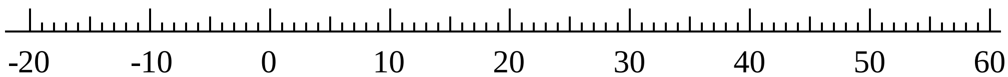
10) $31 =$ _______________$_2$ **11)** $111111_2 =$ ____ **12)** $2^6 - 1 =$ __________$_2$ = ____$_{ten}$

***13)** $11111_2 = 2^{----} - 1 =$ _____ **14)** $1111_2 =$ ____$_{ten}$

****15)** $11111111111111111111111111111111_2 = 2^{-------} - 1 =$ ______________________$_{ten}$

calculator okay

QUIZ: NUMBER LINE JUMPS — Units 5 - 7



Number line for reference if needed

Recall that a 10n + 4 number is 14, 24, 104, etc., a number 4 more than a multiple of 10.

1) What are the next four 10n + 3 numbers after 35? ____________________

2) Circle all the 8n + 3 numbers: 33, 43, 57, 67, 71, 91, 8019, 28019, 28099.

3) T/F: An 8n + 4 number, when divided by 8 will have remainder 4. ____

4) 93 is an 8n + ___ number.

5) A +8 jumper starts on 100. Tell all numbers hit before 136. __________________________

6) A +6 jumper and a +8 jumper both start from the same point. How far apart will they be after they each have made 11 jumps? _____

7) A –4 jumper starts on 12.What are the first 4 negative numbers it will hit? _____________

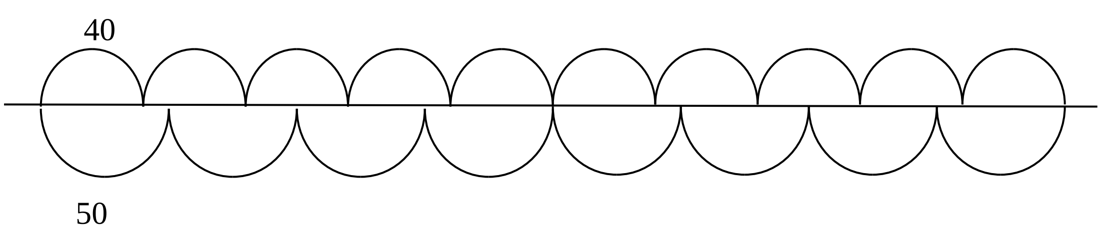
8) A –7 jumper starts on 12. What are the first four negatives it will hit? _________________

9) A + 5 jumper starts on –32. What are the next 4 points it will hit? ____________________

10) A –8 jumper starts on –5601. What are the next 4 points hit? _______________________

11) A +6 jumper and a + 4 jumper both start on –20. Name the next four common landing points.

____



Notice that the ratio of the size of the upper arc to the size of the lower is 4 to 5.

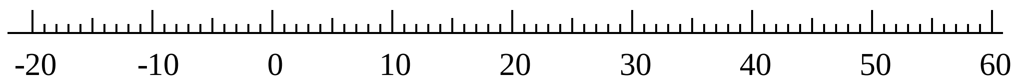
12) Circle the *letter* of each of the following pairs of numbers that could replace the 40 and 50; that is, have the ratio 4 to 5 (or 4:5),

a) 2 to 2 ½ **b)** .4 to .5 **c)** 4 billion to 5 billion **d)** 4×10^{16} to 5×10^{16}

***e)** 4^{10} to 5^{10} **f)** 10^4 to 10^5 **g)** 2.4 to 3 **h)** 8/17 to 10/17 **i)** 3/8 to 3/10

j) 10 + 4 to 10 + 5 **k)** 45 to 55 **l)** 45 to 54

QUIZ: NUMBER LINE JUMPS Units 5 - 7



13)

What is the LCM (lowest common multiple) of:

a) 2, 3, 6?___ **b)** 2 x 1000, 4 x 1000, 6 x 1000?___________

c) 10^4, 10^6 _________ **d)** 3 x 10^4, 7 x 10^5 ____________________________

14) A jumper starts on 30. Name the next 4 landing points hit if it is a:

a) –3 jumper ______________ **b)** – 4 jumper ________________ **c)** – 5 jumper __________

***15)** Name all common multiples of 3 <u>and</u> 4 between 58 and 134.

____

> A number is a multiple of 3 if the sum of its digits is a multiple of 3. A number is a multiple of 4 if its last 2 digits form a multiple of 4.

16) Name all the common multiples of 3, 4 and 10 between 598 and 1001. ____

17) Circle each relatively prime pair (sharing <u>no</u> factors greater than 1).

4,6 4,16 4,21 111, 300 6,39 9,29 9,39 13,39 *69, 46

> A number is a multiple of 9 if the sum of its digits is a multiple of 9.

18) If 3 is a factor of 69, what is its "partner factor"?_____

19) T/F: A common multiple of 6 x 10^5 and 8 x 10^2 *must* be:

a) A multiple of 48 ___ **b)** A multiple of 10^{10}___ **c)** A multiple of 10^7 ___

d) A multiple of 24 ___ **e)** A multiple of 10,000,000,000 ___

20) T/F: The product of two relatively prime numbers is their LCM ___

21) Is every pair of consecutive numbers relatively prime?______

22) Let [8, 20] represent the interval on the number line 8 to 20 inclusive (including 8 and 20).

Let (8, 20) represent the interval on the number line 8 to 20 <u>ex</u>clusive (excluding 8 and 20).

How many multiples of 4 are in the interval (20, 28) __, (20, 28] __, [20, 28] __, [20, 28) __.

23) The difference between two natural numbers is always/sometimes their GCF ____________.

24) T/F: It is always true that the product of two natural numbers is equal to the product of their GFC x LCM. _____

***25)** If the product of two numbers is 864 and their GCF is 24, what is their LCM? ______

QUIZ: NINES — Unit 8

Mentally is best

1) 900 x 80 = __________ **2)** 90 x 700 = _____ **3)** 90 x 4000 = 36 x $10^{\square}$

4) 900 x 6000 = _____ x $10^{\square}$ **5)** 80 x ____________ = 72 x 10^6 (no exponent)

6) 9 x 120 = _____ **7)** 4 x 9 x 10^3 = ______________ **8)** 9900 = 9 x ______

9) 9 x ___ x $10^{\square}$ = 10,800 **10)** ____ x 9 x $10^{\square}$ = 81 x 10^7

For examples 11) to 14), write only the final digit of each number to be multiplied.

11) 9 x 478 = __ **12)** 9 x 612 =__ **13)** 9 x 687 = __ **14)** 9 x 34206 = __

15) 90 x 10^{10} x 12000 = ____ x $10^{\square}$

16) 11 x 9000000 = ________________________________

17) 108 ÷ 9 = ______ **18)** 180000 ÷ 9 = _____________

19) 10,800,000 ÷ 90 =______________ **20)** 900 x 80 = _________

21) _____ x 9 = 5400 **22)** 450000 ÷ ______ = 90 **23)** $\frac{1}{\square}$ x 630 = 70

24) 5/8 x 72 = ___ **25)** 8/9 x $\square$ = 96

26) List all the nine fact answers

9
18
___
___
___
___
___
___
___
___
___
___

27) List all the nine fact answers starting w/108. Watch for patterns.

108
___
___
___
___
___
___
___
___
___
___
___
___

28) A number is divisible (evenly) by 9 if the sum of its digits is divisible by 9. Circle every such number.

1008 99999999 909693999
99900091818 123123123123
9^{345} 45 45^2 4005^9 9^{4005}
9^{4004} 9^{10} 10^9 (6 + 9 + 3)
9^{31} 3^2 3x3x3x3x3
9 x 3 x 3 x 3 27^4

QUIZ: NEGATIVE NUMBERS

Unit 9

Most of us carry a number line around in our head for certain mental work. Use yours if you have one. Answer NF if the question is not fair without calculator.

1a) 90 + (+10) = ___ **b)** 90 + (-10) = ___ **c)** 90 – (+10) = ___ **d)** 90 – (-10) = ___

2a) -90 + 10 = ___ **b)** -90 – 10 = ___ **c)** -90 + -10 = ___ **d)** -90 – (-10) = ___

3a) 60 – (+7) = ___ **b)** 60 – (-7) = ___ **c)** 60 +(+7) = ___ **d)** 60 + (-7) = ___

4a) 40 – (2 x -4) = ___ **b)** 40 + (2 x 4) = ___ **c)** 40 – (2 x 4) = ___ **d)** 40 + 2 x (-4) = ___

5a) 16 – (-3) = ___ **b)** 4000 – (-4000) = ___ **c)** 3000 + (-3000) = ___

6) 10^4 – (-4) = ________

7) -4 – 10^4 = _________

8) 4.8 – 2.3 = ___

9) 10.8 – (-2.5) = ___

10) (7 x 10^3) – (4 x 10^3) = ________

11) (4 x 10^2) – (7 x 10^3) =_______

12) (8 x 10^2) – (3 x 10^3) = ________

QUIZ: NEGATIVE EXPONENTS

Unit 10

1a) For any numbers a, b, and c, (a + b) + c = a + (b + c). The name of this property is the

__________________ property for addition.

b) State this property for the other operation for which it is true: "For any numbers a, b, & c ____"

2) T/F: (80456 ÷ 30934) ÷ 326 = 80456 ÷ (30934 ÷ 326) ____

3) $60.0147 \times 10^{\square} = 6001470$

4) $372.61 \div 10^3 =$ ______

5) $2056.483 \times 10^3 \div 10^5 =$ ____________

6) $2056.483 \div 10^2 =$ ____________

7) $20.56 \times 10^5 =$ ___________

8) $20.56 \div 10^5 =$ _______________

9) $20.56 \times 10^{-5} =$ _____

10) $56.89 \times 10^2 =$ _______

11) $56.89 \times 10^{-2} =$ ________

12) $32.6 \times 10^{\square} = 3.26$

13) $32.6 \times 10^{\square} = 326$

***14)** $32.6 \times 10^{\square} = 32.6$

15) Double Opposites:

a) $\begin{cases} 23 - (+5) = \_\_\_ \\ 23 + (-5) = \_\_\_ \end{cases}$

b) $\begin{cases} 12 \div \frac{2}{3} = \_\_\_ \\ 12 \times \frac{3}{2} = \_\_\_ \end{cases}$

c) $\begin{cases} 21.34 \div 10^3 \,\_\_\_\_\_\_\_\_\_\_ \\ 21.34 \times 10^{-3} \,\_\_\_\_\_\_\_\_\_\_ \end{cases}$

16) What is the additive opposite of -6 ? ___

17) What is the multiplicative opposite of 5? ___

18) A number multiplied by its multiplicative opposite = __.

19) The reciprocal of 3/7 is ____; of 5 is ____.

20) In 18 + 0, the additive ***identity*** is ___ because it allows 18 to retain its _________.

21) What is the reciprocal of 0? ______________________________

22) 163.8 times its multiplicative opposite is ___.

QUIZ: NEGATIVE EXPONENTS — Unit 10

As in Unit 10, Op means additive opposite (-) and R means reciprocal: R 5/6 = 6/5

T/F:

23a) Op 5 = -5 ___ **b)** R 5 = 1/5 ___ **c)** –(-(-(17) = 17 __

d) $10^3 \times 10^{-3} = 10^0$ ___ **e)** $10^0 = 1$ ___ **f)** R 3/5 = 5/3___

g) –(-(-(-(-(-12))))) = 12 ___ **h)** R 12 = 1/-12 ___

Let n represent any number except 0.

i) R n = 1/n ___ **j)** Op n = -n ___ **k)** Op (–n) = n ___

l) R 1/8 = 8 ___ **m)** (R 7) x 7 = 1 ___ **n)** $7^{-1} \times 7^1 = 1$ ___

o) $7^{-1} \times 7^1 = 7^0$ ____ **p)** $\frac{7^{23}}{7^{23}} = 1$ ____ **q)** $\frac{7^{52}}{7^{52}} = 7^0$ ____

r) $7^0 = 1$ ____ **s)** $5^0 = 1$ ____ **t)** $n^0 = 1$ ____$(n \neq 0)$

u) $6^3 \div 6^4 = 6^{-1}$___ **v)** $6^{38} \div 6^{39} = 1/6$ ___ **w)** $1/6 = 6^{-1}$ ___

x) $6 = 6^1$ ___ **y)** $6^1 \times 6^{-1} = 0$ ___ **z)** $n^{-1} \times n^1 = 1$ ___

aa) R $6^{-1} = 6$ ___ **bb)** R $\frac{2}{3} = 1/2/3$ ___

cc) n + (-n) = n – n = 0 ___ **dd)** n x (R n) = 1 ___

There is no quiz for Unit 11

QUIZ: A GLIMPSE INTO THE INFINITE — Unit 12

1) What is the name of the matching process often used in kindergarten to compare the number of objects in one set to the number in another? ____

2) $\aleph_0$ is called "Aleph Null". Aleph, being the name of the first letter in the Hebrew Alphabet, is the name given to the number of natural numbers 1, 2, 3, 4, etc. by Georg Cantor. This assignment of $\aleph_0$ to the set of natural numbers suggests that we are dealing with the _________ (first, only) infinite set of numbers that exists.

3) Suppose that you "make up" a set and call it set M. Let M = $\{r, s, t, u\}$. **Is** $\{r, s\}$ a proper subset of set M? _____

4) Equal vs. equivalent.

a) T/F: $\{a, b, c, d\}$ and $\{b, a, d, c\}$ are equal. ___ **b)** T/F: $\{a, b, c, d\} = \{a, b, c, d\}$ ____

c) T/F: $\{1, 2, 3, 4\}$ equals $\{a, b, c, d\}$ ____ **d)** T/F: $\{1, 2, 3, 4\}$ is equivalent to $\{a, b, c, d\}$ ____

5) T/F: The even and odd numbers are equivalent. ____

6) T/F: The even and odd numbers are in 1 to 1 correspondence.____

7) A = $\{1, 2, 3, 4, 5, 6, \ldots\}$

B = $\{0, 2, 6, 12, 20, 30, \ldots\}$

a) T/F: For every number in set A there will always be a corresponding number in set B, <u>and</u> for every number in set B there will be a corresponding number in set A. If not, the two sets are not in 1 to 1 correspondence. ____

***b)** For 100 in set A, what is its corresponding number in set B? _____

****c)** For 930 in set B, what is its corresponding number in set A? _____

*****d)** For 160,400 in Set B, what is its corresponding number in set A? ________

***e)** The definition of an infinite set is ____

____

QUIZ: DECEPTIVE RATES

Unit 13

1) If Ernie can spray paint a bike frame in 24 minutes and Bernie can do it in 30 minutes, how long will it take them to do it working together? This is an excellent problem to give to anyone who has not had second year algebra recently if you want to win a bet that they cannot do it.

a) So Ernie's spray painting rate of 24 *minutes* per bike can be called ____ (reduced fraction) *hour*/bike.

b) Bernie's slightly slower rate of 30 minutes per frame, can be called ___ (reduced) hour per bike.

c) Using the fractional expression in each case , what is the reciprocal of each rate?

I) Ernie's reciprocal rate: _____ bikes per hour

II) Bernie's reciprocal rate: _____ bikes/hour

d) If Ernie and Bernie work together on the same bike, each without suffering a slowing in his rate and without spraying each other, how many bike(s) will they spray in 1 hour? _______. (This is not a difficult question if you realize that you already have their individual rates for bikes in one hour)

e) Your answer to d) is the number of bikes they can spray per hour together, but the underlined question in the first paragraph wants the opposite of that: not bikes per hour, but hours per bike. You now *have* the bikes per hour rate but from that you must find its ______ ____ _____rate which is _____ (fraction) hour for one bike when Bernie and Ernie work together.

2) Take the reciprocals of the original fraction rates for Ernie and Bernie, add those reciprocals, and then take the reciprocal of that answer. Briefly, find the reciprocal of the sum of their reciprocals. With a little luck you should get a familiar answer. __________

3) Stacy and Stacey work at a kennel where they make their own doghouses. Stacy takes 40 minutes to paint a new doghouse and Stacey takes 50 minutes. How long will it take them to paint a doghouse working together?

***4)** Bob, Rob and Palo can fill a watering trough using buckets from a nearby shallow well. Bob takes 1/2 hour, Rob 1/4 hour and Palo 10 minutes. . How long will it take the three boys to fill the tank working together? ____

____

QUIZ: TIDAS AND SERIES — Units 14 - 16

From unit pages:

a) Number of terms (n)	1	2	3	4	5	6	__	__	__	__
b) Sum of first n odd numbers (n^2)	1	4	9	16	25	36	___	___	___	____
c) Sum of first n even numbers ($n^2 + n$)	2	6	12	20	30	42	___	___	___	____
d) Sum of first n consecutive natural numbers	1	3	6	10	15	21	___	___	___	____

1) You learned to construct the top row in or before grade 1. Fill its blanks now in a.

2) Tell how each number in b. grows from the one before it. (You already know how each number is the square of the number above it.) ____

3a) In **c**, why does it say ($n^2 + n$)? ____

b) Finish the row and **tell** where that comes from? ______________________________________

c) Notice how row c's numbers grow from left to right and fill its blanks if you have not already done so.

4) In the chart above there is no formula nor any starting totals for sums of the natural numbers. Your task is to find the sum of the first 6000 natural numbers.____________________.

5) Find the sum of the natural numbers from 6001 to 8000.________

____

QUIZ: SETS AND UPSETS — Unit 17

1) Exactly which sets have the empty set as a subset. ____________________________

2) Another word for "empty" in the term "empty set" is _______________

3) T/F: $\{5, 6, 8\} \subseteq \{5, 6, 8\}$ ___

4a) T/F: $5, 6, 8 \subset \{5, 6, 8\}$ ___ **b)** T/F: $6 \subset \{3, 6, 8\}$ ___ **c)** T/F: $6 \in \{5, 6, 8\}$ ___

5) Give the power set of $\{t, u, v\}$. $\{$____$\}$

6) How many sets in the answer to exercise 5) ? ____

7) How many subsets are in the set of $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ ________

8) How many members are in the power set of a set with 10 elements? _____

9) How many subsets are in a set with 250 members? Use an exponent in your answer.____

***10)** How many members in a set which has 256 members in its power set? ______

11) T/F: a) $\{7\} \in \{7\}$ ___ b) $\{7\} \in \{7, 8\}$ ___ c) $\{7\} \subseteq \{7, 8\}$ ___

12) T/F: a) The natural numbers include zero. ___

b) The whole numbers include zero ___

c) The integers include zero ___

d) There are two sets here $\{\varnothing\}$ <u>and</u> exactly one is empty. ___

13) Let A = $\{2, 5, \{1, 3\}\}$, Let B = $\{1, 3\}$ and let C = $\{3\}$.

T/F: **a)** $2 \subset A$ ___ **b)** $B \subset A$ ___ **c)** $3 \subset C$ ___ **d)** $1 \in A$ ___

14) Fill in every correct answer among the choices. Use the sets in exercise 13.

a) ____________________ is a member of set B.

b) ____________________ is a subset of set B. (Note that $\{3\}$ is not a subset of A.

c) ____________________ is a subset of A, B <u>and</u> C.

d) ____________________is a of member A.

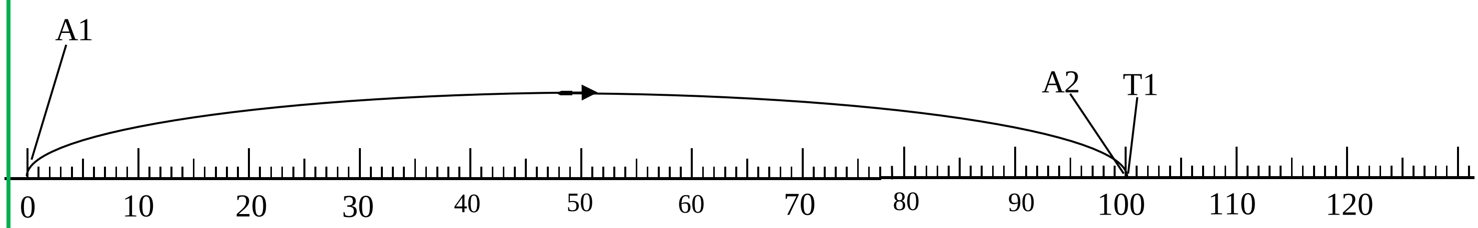
e) How many sets are in the power set of set A? ____

***f)** Name the entire power set of Set A. Get every set.

$\{$____$\}$

QUIZ: ACHILLES

Unit 18



Recall that Achilles could run at ten times the speed of the tortoise, and that he gave the tortoise a 100 meter head start. A1 designates Achilles' starting position, and T1 is the tortoise's first position.

In chasing the tortoise, while Achilles ran to the tortoise's start position, T1 (and A2), the tortoise meanwhile ran 1/10th of that distance because Achilles runs 10 times as fast.

1) Carefully draw straight lines to positions on number line from the remaining A and T designations.

2) Fill the blanks in this chart which gives the numerical value of the landing points.

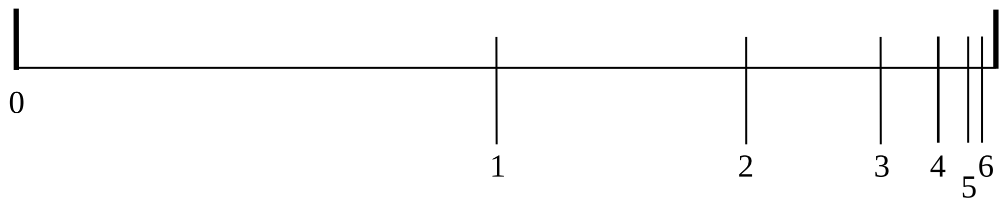
				Difference, T minus A
A1---	0	T1---	100	100
A2---	100	T2---	110	10
A3---	______	T3---	______	______
A4---	______	T4---	______	______
A5---	______	T5---	______	______
A6---		T5---		______
				______
				______
				______

3) Notice that Achilles starts at 0 and the tortoise, starts at______. After that, they have the same landing points, but are never at the *same place at the same time*. Complete the chart down to the bottom of each column, keeping decimal points lined up when they appear. The "Difference" column shows that Achilles gets closer and closer to the tortoise. If you added on a thousand lines below T3, the difference between Achilles and the tortoise would be (circle correct answer): 1/100000, 10^{-6}, 10^{-1000}

4) We have now the idea of Achilles drawing closer and closer to the tortoise, apparently without an ending. We can see from the number line and the T4, T5, . . . numbers on the chart that the tortoise jumps will total (fill in whole number part) ____.11111111111 . . . . We know from unit 18 that 1/9 = $9\overline{)1.00000000}$ = .1111111111111111111 . . . Therefore, the precise measure of the distance traveled by the tortoise when overtaken is ___________________.

QUIZ: HALFWALK

Unit 19



1) T/F: In Achilles, the distances of the tortoise and Achilles were measured in meters, while the distances in Halfwalk were measured in fractional parts of the room width. ___

2) The numbering system above is even simpler but doesn't give much information about a vertical line aside from its order: "This is the 3rd line and that is the 5th line", for example. But suppose we use these counting numbers to correspond to fractional parts in <u>Halfwalk:</u> (List the next 3 fractions.)

not fifty-six

1	2	3	4 56		
1/2,	1/4,	1/8,	_____,	____,	_____

3) Write the fraction which corresponds to 8:____; to10. _____; to 100 (Wait! See exercise 4.)

4) You might have seen *how* the numbers and fractions <u>correspond:</u> 2 with $1/2^2$, 3 with $1/2^3$, etc.

a) 10 with $1/2^{10}$ =1/1024, 100 with $1/2^{\boxed{}}$ = 1/1267650600228229401496703205376.

(Computer's calculator)

5) Use your calculator to find the <u>fractions</u> corresponding with

a) 19_____________ **b)** 29 _____________ **c)** 86 Use exponent: _______

6) The accumulated distances look like this. Give the distances for **(4), (5) and (6)**:

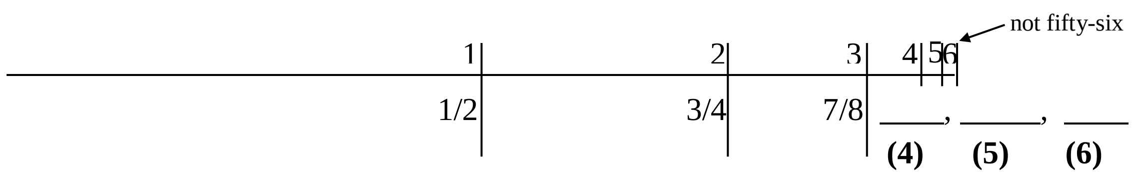


Figure 2

QUIZ: HALFWALK

Unit 19

7) The correct answer to the distance corresponding with (6) is 63/64. In this infinite series similar to Halfwalk, the number of the term n, is 6, the denominator is 2^6, and the numerator is 1 less than the denominator, or 2^6 – ___.The 11th term is $\frac{2^{--}-1}{2^{---}}$, which equals:_______ _______ (Fraction with whole numbers, top and bottom).

8) Continue these fractions to12, corresponding to counting numbers as in Figure 1, above.

1/2, 1/4, 1/8, 1/16,____

9) Continue these fractions as in Figure 2, above, to the counting number 13:

127/128, 255/256,____

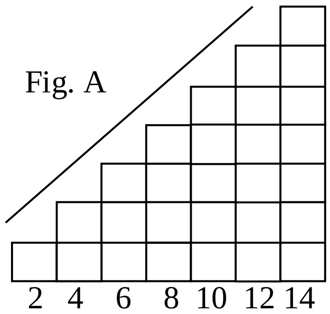
****10) a)** $\frac{3}{2^?}+\frac{1}{8}=\frac{1}{2}$, $?=$ ___ ****b)** $\frac{20}{2048}-\frac{\square}{512}=\frac{1}{512}$ ****c)** $\frac{53}{1024}-\frac{5}{128}=\frac{13}{2^?}$, $?=$ ___

There are no quizzes for 20 and 21.

QUIZ: CARL FRIEDRICH GAUSS

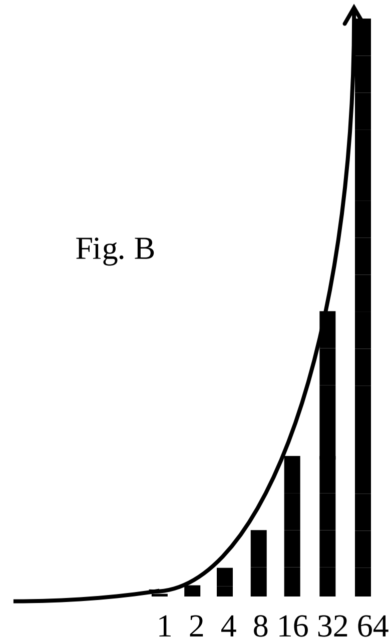
Unit 22

Fig. A



Summing series can be done in a variety of ways if we have a ”linear” series. “Linear” contains the word line, and in this case suggests straight line, at left

Fig. B



1) You can tell the average of the heights above (Fig. A) almost at a glance, because the increase is *linear* and also because there is an odd number of numbers, so there is a middle number, 8. Is 8 the average height of the stacks? ____

2) What is the sum of these 7 even numbers? _____

3) Did you multiply the average height times the number of stacks? _____ Of course you did.

4) Over in Fig. B we also have a middle number. It is _____. Is 8 the average height of the black rods? ______. In the above diagram of 7 rods the sum of any group starting at 1, say 1 + 2 + 4 + 8 is 1 less than the next number, 15.

5) What is the sum of 1 + 2 + 4 +. . . + 128?_______ Of course, this is a reversed view of the base two number system as you saw in Unit 4: 16, 8, 4, 2, 1.

6) Give the binary number version (ones and zeros) of:

a) 128 __________ b) The answer to exercise 5 ___________

QUIZ: CARL FRIEDRICH GAUSS

Unit 22

7) How many terms are in the series 643 + 651 + 659 + . . . + 891 + 899? (Put answer in c).

Hints:

a) Because these are separated by 8, they are 8n + ___ numbers.

b) The number of terms will remain the same if we subtract 3 from each, and then we can divide each term by ___.

c) The sum of the resulting series will be consecutive but not starting with 1. Fix that by subtraction and you will see that the number of terms is ____.

d) The series sum is ________.

8) T/F: The average of any *linear* series (see above) is (1^{st} term + last term)/2 ___

9) Gauss' problem was to find the sum of the numbers from 1 to 200. The procedure 201 x 200 x ½ = _________ will fit both procedures; (1) Average x Number of terms, and (2) Gauss' idea; series + backwards series, multiply by No. of terms/2. (T/F)____

10) Find the sum of the series 103 + 110 + 117 +. . . + 873 + 880. ______.

QUIZ: ORDER OF OPERATIONS — Unit 23

1) $12 \div 6 \div 2$ could have two values. Name them and then pick the right one.__________.
Suggestion: Change the divisions to multiplication of opposites (reciprocals). That would be 12 x 1/6 x ½ = __. The answer of 1 is correct. Did you get it? This tells us that $12 \div 6 \div 2$ should be done from left to right; $(12 \div 6) \div 2 = 1$. It also tells us that, unlike multiplication, *division is not associative*. It matters how you *associate* the numbers.

2) Give two answers, a) the correct answer, and b) the most likely wrong answer: 12 – 6 – 2

a) ___________ **b)** _________

3a) Is subtraction associative? ____ **b)** Multiplication? ____ **c)** Addition? ____

4) Try $12 \div 4 \times 3 =$ ____

5) $12 \div 4 + 3 =$ ____

6) Which is correct, **a)** $\dfrac{6+\cancel{4}^{\,2}}{\cancel{2}_{\,1}} = 8$ or **b)** $\dfrac{\cancel{6}^{\,3}+\cancel{4}^{\,2}}{\cancel{2}_{\,1}} = 5$? ______

The wrong answer is quite common.

7) Recall that 5(3+7) means 5 x (3 + 7). Use the expression at right to practice the correct order of these operations. Correctly order list below by writing their letters here: ↙

$$\frac{2(2^3-3)^2}{8^2-2^2} = 5/6$$

c) Perform additions and subtractions ___ ___ ___

b) Apply exponents

a) Perform multiplications and divisions

Keep work organized:

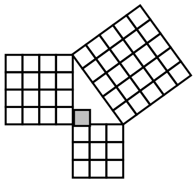
> We use common sense with parentheses and combine what is inside them before going on. See above fraction.

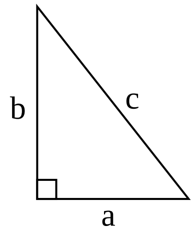
8) $3\left(5\sqrt{25} + 5(3^3 - 4 \times 6)\right)^2 =$

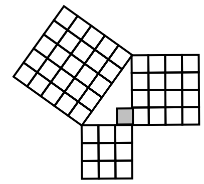
9) $48 \times \left(\dfrac{3}{4}\right)^2 + 3(1508 - 1499)^2 =$

10) $\left(\{(10-3)+13\}^2 + \{(8+1)^2 - 8 \times 10\}\right)^2 =$

QUIZ: PYTHAGORAS Unit 24

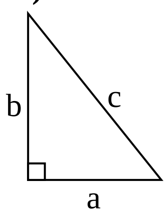






$a^2 + b^2 = c^2$

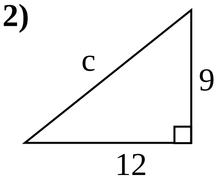
1)



a = 3
b = 4
c = ?

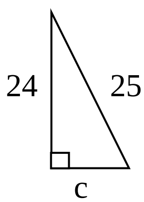
$3^2 + 4^2 = c^2$
$9 + 16 = c^2$
$25 = c^2$
$c = \sqrt{25}$
c = ___

2)



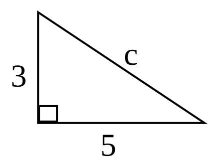
$9^2 + 12^2 = c^2$
____ ____ $= c^2$
____ $= c^2$
$c = \sqrt{225}$
c= ____

3)



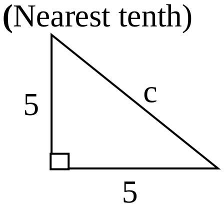
c = ___

4) (Nearest tenth)



c = _____

5) (Nearest tenth)



c = ___

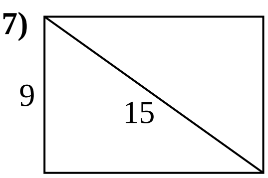
Use calculator only if necessary.

6a) $\sqrt{400} =$ _____ **b)** $\sqrt{90000} =$ ______ **c)** $\sqrt{14400000000} =$ __________ ***d)** $\sqrt{.90} =$ ____

e) $\sqrt{.900} =$ _____ **f)** $\sqrt{.0009} =$ ____ **g)** $\sqrt{9} \times \sqrt{9} =$ ___ **h)** $\sqrt{8} \times \sqrt{8} =$ ____

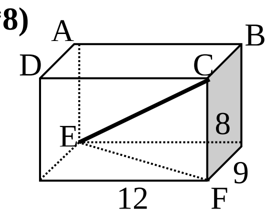
i) $\sqrt{13.4} \times \sqrt{13.4} =$ ______ ***j)** $\boxed{}^2 = 6$ No decimal, exact answer. **k)** $\sqrt{57^2} =$ ____ **l)** $\left(\sqrt{}\right)^2 = 7$

7)



Find the area of a rectangle whose height is 9 in. and diagonal is 15 in.

****8)**



The **bold segment** EC is called a main diagonal of the prism (box). ABCD is the prism top, 8 is its height, 12 is its length, and 9 its width. Find the length of EF and then find the measure of EC.______

There is no quiz for Unit 25

QUIZ: MULTIPLYING BY 12 — Unit 26

Do Mentally

1) 12 x 10 = ____ **2)** 12 X 5 = ____ **3)**120 x 5 = ________ **4)** 1200 x 50 = _____

5) 10 x 50 = ____ **6)** 9 x 12 = ______ **7)** 90 x 20 = ________ **8)** ___ x 12 = 10,800

9) 6 x 12 = ____ **10)** 120 x 120 = _______ **11)** ________ x 60 = 72,000,000

12) 4 x 12 = ____ **13)** 8 x ______ = 9600 **14)** 12 x 12 = ______ **15)** 11 x 12 = _____

16) 11 x .12 = _____ **17)** 12^2 x 2=______ **18)** 12 x 80 = ____ **19)** 1.2 x 80 = ____

20) 12 x 8 = ____ **21)** 7 x 12 = _____ **22)** 12 x 2^{----} = 96 **23)** 12 x 3 = ____

24) 12 x 2^2 = ____ **25)** 12 x 3^2 = ____ **26)** 12 x 18 = _____ **27)** 12 x 7 = ____

28) 12 x 14 = ____ **29)** 140 x 120 = ____ **30)** 14 x 120 = _____ **31)** 12^2 x 10 = ______

32) 7 x 1200 = ______ **33)** 60 x 70 = ______ **34)** 12^2 x 10^3 = ______ **35)** 10 x 12 = ______

36) 11 x 12 = _____ **37)** 12 x 11 = _____

***38)** 11 x 11 = ____ This → 11 / x11 can be done mentally

11
11
121

with the cloud as the mentally computed answer.

***39a)** Try mentally 11 x 111 = _______________ ***39b)** 111 x 111 = _____________

40) 12 x 12 x ½ = 12 x ___ = ____ **41)** 12^2 x 1/3 = 12 x 12 x 1/3 = 12 x ___ = ____

Do as in exercise 38:
***42)** 11 x 13 =_______

43) $\sqrt{12} \times \sqrt{12}$ = ____ **44)** $\sqrt{144}$ x 12 = _____ **45)** 12 x $\sqrt{49}$ = ____

46) $\sqrt{12 \times 300}$ =____

Do as in exercise 38:
***47)** 11 x 22 = ______

***48)** 22 x 22 = _____

QUIZ: SOME OLD, SOME NEW, FRACTIONS, DECIMALS AND PERCENTS

Unit 27

1) 1/4 = ____% **2)** 1/2 of 25% = ____% **3)** 3/4 = ____%

4) Halfway from 50 to 75 = ____. **5)** Halfway from 1/2 to 3/4 = ____

6) 5/8 = ____% **7)** 1/8 = ____% **8)** 1/10 = ____% **9)** 1/5 = ____ % **10)** 3/5 = ___%

11) 4/5 = ____% **12)** 50% = ____(fraction) **13)** 12 ½% = ____(fraction)

14) 50% + 12 ½% = ____ ***15)** 1/2 + 12 ½% = ____% = ____(fraction)

16) 1/2 of 12 ½ % = ____% = ____(fraction). **17)** 25% x 40 = ____ **18)** 100% of 40 = ____

19) 125% of 40 = ____ **20)** 112 ½ % of 8 = ____ **21)** 6 ¼ % of 8 = ____ **22)** 1/3 = ____(decimal)

23) 1/3 x .999 ⋯ = ____ **24)** 1/3 = _______%(exact) **25)** 2/3 = ____%

26) 5/9 = ____ (decimal) **27)** $.33\overline{3}$ = ____(fraction) **28)** 1/3 = ._ _ _ ⋯

29) 2/3 = . _ _ _ ⋯ **30)** 1/3 + 2/3 = ____ **31)** .33⋯ + .66 ⋯ = .999 ⋯ = 1 (T/F) ____

32) .02 = ____(fraction) **33)** .03 = ____(fraction) **34)** .03 ½ = ____(decimal)

35) .03 ¼ = ________(decimal) **36)** .03 ¾ = _________(decimal)

To fractions: **37)** $.\overline{142857}$ = ____ **38)** $.\overline{857142}$ = ____ **39)** $.0\overline{571428}$ = ____

40) .075 = ____ (fraction) ***41)** What is the reciprocal of .0375? _____ (fraction)

42) $.\overline{285714} + .\overline{714285}$ = .999999 (T/F) ___ **43)** $.\overline{285714} + .\overline{714285}$ = 1 (T/F) ___

QUIZ: SQUARES AND SHORTCUTS

Unit 28

Do mentally

1) $8 \times 2 = 4^{---}$ **2)** $8^2 = 4^{----}$ **3)** $8^2 = 2^{----}$ **4)** $10^2 =$ _____ **5)** $11^2 =$ _____ **6)** $12^2 =$ _____

If 12 x 12 = 144, then
13 x 12 = one 12 more than 144.

7) 13 x 12 = _____ **8)** 13 x 13 = _____ **9)** 14 x 14 = _____ (Almost 200) **10)** 14 x 15 = _____

11) $15^2 =$ _____ **12)** $(1.5)^2 =$ _____ **13)** $(.15)^2 =$ ______ **14)** $25^2 =$ 6 _ _

15) $(2.5)^2 =$ _____ ***16)** $(1\ 1/2)^2 =$ _____ ***17)** $(2\ 1/2)^2 =$ ____ ***18)** $(3\ 1/2)^2 =$ _____

Notice the pattern in ex. 16, 17, 18, and use it below.

19) $(5\ 1/2)^2 =$ _____ **20)** $(4\ 1/2)^2 =$ _____ **21)** $(9\ 1/2)^2 =$ _____ **22)** $(7\ 1/2)^2 =$ _____

23) $(7.5)^2 =$ ______ **24)** $75^2 =$ ______ **25)** $45^2 =$ ______ **26)** $850^2 =$ ___________

***27)** $(\_\_\_\_\_)^2 = 4225$ ***28)** Find two numbers that make this true: $\square^2 + \square^2 = 221$

29) Sum of numbers to square =19 $\left\{ \begin{array}{l} 9^2 = 81 \\ 10^2 = \_\_\_\_\_ \end{array} \right\}$ Add the 19 to 81.

***30)** Complete as in 28):
$10^2 =$ ____
$11^2 =$ ____

***31)**
$20^2 =$ _____
$21^2 =$ _____
$22^2 =$ _____
$23^2 =$ _____

***32)**
$1000^2 =$ ________________
$1001^2 =$ ________________
$1002^2 =$ ________________

33)
$15^2 =$ ____
$16^2 =$ ____
$17^2 =$ ____
$18^2 =$ ____

35)
10 x 10 = ___
11 x 9 = ___
12 x 8 = ___

36)
20 x 20 = ___
21 x 19 = ___
22 x 18 = ___

37)
29 x 31 = ___
28 x 32 = ___
27 x 33 = ___
26 x 34 = ___

Decide what $\rightarrow$, $\nrightarrow$ tell you to do.

***34a)**	**34b)**
10 $\rightarrow$ 103	10 $\nrightarrow$ 205
20 $\rightarrow$ 403	20 $\nrightarrow$ 805
5 $\rightarrow$ 28	3 $\nrightarrow$ 23
15 $\rightarrow$ ____	30 $\nrightarrow$ ____
16 $\rightarrow$ ____	15 $\nrightarrow$ ____
17 $\rightarrow$ ____	16 $\nrightarrow$ ____
___ $\rightarrow$ 327	21 $\nrightarrow$ ____
___ $\rightarrow$ 487	____ $\nrightarrow$ 37
8 ½ $\rightarrow$ ____	____ $\nrightarrow$ 3205

****38)** In the following, give the pair of factors that are closest together.

a) ____ x ____ = 1599
b) ____ x ____ = 3596
c) ____ x ____ = 4891

QUIZ: MENTAL ARITHMETIC 1 AND 2

Units 29 - 30

1) 20 x 16 = ____________

2) 40 x 16 = ____________

3) 40 x 320 = ___________

4) 5 x 16 = _____________

5) 2 ½ x 160 ____________

6) ¼ of 40 = ____________

(Recall $(8\ ½)^2$ = 72 ¼)

7) $(2\ ½)^2$ = _____________

8) 20 x 1234 = __________

9) 5/8 x 160 = ___________

10) 4/5 x 1000 = _________

11) 99 + 479 = __________

12) 98 + 623 = __________

13) 297 + 148 = _________

14) 39 ½ + 1 ½ = ________

15) 40 – 1 ½ = __________

16) 40 – 2 1/3 = _________

17) 60 – 4 ¼ = __________

18) 7^2 = ______________

19) 2 x 7^2 = ____________

20) $(2 \times 7)^2$ = ___________

21) 4 x 7^2 = __________

22) 8 x 7^2 = ____________

23) 25% of 64 = _________

24) 12 ½% of 64 = _______

25) 6 ¼% of 64 = ________

26) 3 1/8% of 64 = _______

27) 16 x 16 = ___________

28) 15 x 17 = ___________

29) 4 x 18 = ____________

30) 11 x 62 = ___________

31) 2 x 12/47 = _________

32) 11 x 49 = ___________

33) 586 – 101 = _________

34) 586 – 102 ½ = _______

35) 627 – 103 1/3 = ______

36) 1/2 of 12/13 = _______

37) ¼ of 12/47 = ________

38) 2 x 12/47 = __________

39) ½ x 12/47 = _________

40) 2 ½ x 16/47 = ________

41) 1 1/3 of 12 = ________

42) 1 2/3 of 9/100 = _____

43) Does 16 x 25 =

8 x 50? (Y/N) ________

44) 16 x 25 = ___________

45) Does 8 x 412 =

2 x 1648? (Y/N) ______

46) 8 x 410 = ___________

47) 18 x 51 = ___________

48) 16 x 304 = __________

49)1600x1800=__________

50) 120 x 450 = _________

51) 49 x 51 = ___________

52) 11 x $2.99 = _________

53) 148 x 152 = _________

54) 500 x 264 = _________

55) 498 + 672 = _________

56) 101 x 16 = __________

57) 80 – 3 ¾ = __________

58) 160 – 13 3/5 = _______

QUIZ: RECIPROCALS ON THE NUMBER LINE

Unit 31

1) The reciprocal of any negative number is a (negative/positive) _______________ number.

2) The reciprocal of zero is zero (T/F). ___

3) The reciprocal of -5 is +5 (T/F). ___

4) The additive opposite of -5 is +5 (T/F) ___

5) What is the distance between:

a) 10 and its reciprocal? ______

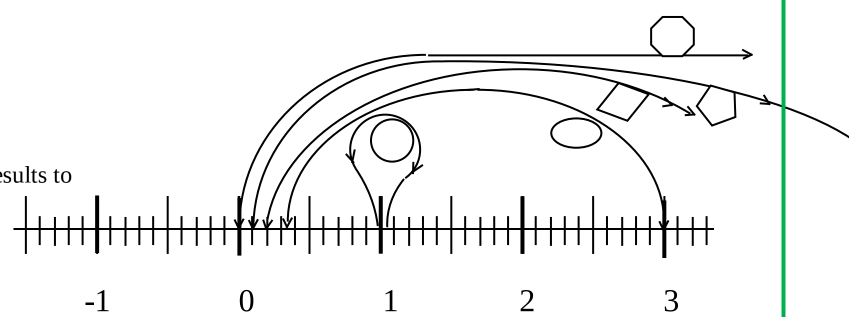
b) 1000 and its reciprocal? __________

c) 5 and its reciprocal? ______

For part **d)** keep results to nearest 100th

d) .98 and its reciprocal? _______

e) $.\overline{3}$ and its reciprocal? ______



6) The outermost curve on the diagram which lies very close to zero, actually lands on what nearby number? __________ (See exercise 5.)

7) Each curve above represents a description in ex. 5. On the diagram, in each shape, print the letter from exercise 5 which belongs in that shape.

8) The reciprocals of each number in the interval from .5 to .75 lie in what interval?___________

***9)** The reciprocals of each number in the interval from 0 to 1 lie where? ___________________

10) What is the reciprocal of 1? ____

QUIZ: A BIGGER INFINITE Unit 32

1) T/F: A rational fraction is one which is, or can be shown to be, a fraction with whole numbers in both numerator (no zero in denominator). ____

2a) 999/1000 is less than 1 by how much? ______ **b)** $\dfrac{17^{640,000}}{17^{640,000}+1}$ is less than 1 by how much? ______

The reciprocal of the expression in b) is (less than, greater than) 1 by how much? ____________

3) Enlarge this array of rational fractions to five rows and five columns. Include arrows.

$$\begin{array}{lll} \frac{1}{1} \rightarrow & \frac{1}{2} & \frac{1}{3} \rightarrow \\ \swarrow & \nearrow & \\ \frac{2}{1} & \frac{2}{2} & \\ \downarrow \nearrow & & \\ \frac{3}{1} & & \\ \downarrow & & \end{array}$$

4a) T/F: The fraction $\dfrac{99}{100}$ is farther from 1 than the fraction $\dfrac{99\frac{1}{2}}{100}$. ___

b) Rewrite $\dfrac{99\frac{3}{4}}{100}$ so it appears as a rational fraction. Hint: Multiply top and bottom by 4.____

c) Yes/No: Is the answer to part b) closer to 1 than either fraction in part a) ? _______

5) Yes/No: Tell whether the following is a <u>fair question</u>:

"Is there a 'next fraction' after 399/400?" ____. Well, is there? ____

6) If you count the rational fractions without eliminating the infinitely many duplicates such as 2/4, 3/6, 100/200, 666/222, etc., will that set still be in 1 to 1 correspondence with the natural numbers?_______

7) The cardinal number of the rational fractions is ___. (The first letter in the Hebrew alphabet)

QUIZ: ARE WE THERE YET? Unit 33

Use your calculator as little as possible while doing this quiz.
There is no need for writing more than one cycle when recording repeating decimals.

1) Change to repeating decimals such as $.\overline{135}$.

a) $\frac{1}{9}$ = ___ **b)** $\frac{7}{9}$ = ___ **c)** $\frac{7}{99}$ = _____ **d)** $\frac{7}{990} = (\frac{7}{99} \text{ x } \frac{1}{10}) = .\overline{07} \text{ x } .1 =$ _____

***e)** $\frac{8}{9900}$ = _______ **f)** $\frac{70}{99}$ = _______ ***g)** $\frac{13}{99}$ = ______ ***h)** $\frac{13}{990}$ = ________

i) $\frac{1}{7}$ = _____________ **j)** $\frac{3}{7}$ = _____________ **k)** $\frac{3}{11}$ = ____ ***l)** $\frac{6}{70}$ = _____________ Compare g) and h)

m) $\frac{1}{11}$ = ______ **n)** $\frac{7}{11}$ = _______ **o)** $\frac{7}{90}$ = ______

2) Change to <u>reduced</u> fractions, or mixed numbers like $2\frac{1}{7}$.

a) $.\overline{4}$ = ___ **b)** $.\overline{231}$ = _____ **c)** $3.\overline{03}$ = ______ ***d)** $0.0\overline{03}$ = ________

e) $.\overline{2} + .\overline{7} = 1$ exactly (T/F)__. **f)** $.\overline{285714}$ = _____ ***g)** $.00\overline{1}$ = _____

***h)** $.00\overline{428571}$ = _______ **i)** $3.00\overline{857142}$ = ______

3) Fill in the blanks:

Integers {
Negative whole numbers
____
____

_____________ **Numbers** {
____
Ratios of one integer to another

_____________ **Numbers** {
___________ ____________
Solutions to ____________ _____________

4a) T/F: Every repeating decimal is a "rational fraction" and have the cardinal number $\aleph_0$. ___

b) T/F: Cantor showed that the set of "irrationals" like $\sqrt{2}$ (non-repeating) that are solutions to algebraic equations, has $\aleph_0$ members __ **c) T/F:** Cantor thus showed that the set of all irrationals is in 1 to 1 correspondence with the naturals.___

QUIZ: MENTAL MATH 3, FRACTIONS — Unit 34

Do Mentally

a) $\frac{1}{8}+\frac{1}{100}=\frac{108}{800}=\frac{27}{200}$ b) $\frac{1}{7}+\frac{3}{100}=\frac{121}{700}$ c) $\frac{2}{7}+\frac{5}{10}=\frac{55}{70}=\frac{11}{14}$

Samples for study of patterns

d) $\frac{12}{100}-\frac{1}{12}=\frac{44}{1200}=\frac{11}{300}$ e) $\frac{7}{10}-\frac{3}{7}=\frac{19}{70}$

1) $\frac{1}{7}+\frac{1}{6}=\frac{\quad}{42}$ **2)** $\frac{1}{10}+\frac{1}{11}=\frac{21}{\quad}$ **3)** $\frac{1}{400}+\frac{1}{401}=$ **4)** $\frac{1}{100}+\frac{3}{4}=$

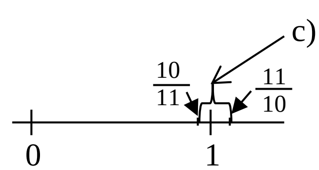
5) $\frac{1}{29}+\frac{1}{31}=$ **6)** $\frac{1}{9}+\frac{3}{8}=$ **7)** $\frac{5}{18}+\frac{22}{16}=$ **8)** $\frac{3\frac{1}{2}}{12}+\frac{2}{3\frac{1}{2}}=$

Lots of mental shortcuts here.

9) $\frac{1}{28}-\frac{1}{32}=$ **10)** $\frac{1}{27}-\frac{1}{33}=$ **11)** $\frac{3}{10}-\frac{7}{40}=$ **12)** $\frac{2}{\square}+\frac{3}{20}=\frac{\bigcirc}{60}$

13) $\frac{\square}{6}+\frac{4}{\bigcirc}=\frac{49}{30}$ **14)** $\frac{\square}{100}+\frac{3}{\bigcirc}=\frac{307}{700}$ **15)** $\frac{3}{2}-\frac{2}{3}=$ **16)** T/F: $\frac{7}{6}-\frac{6}{7}=\frac{1}{6}+\frac{1}{7}$ ___

17a) $\frac{11}{10}-\frac{10}{11}=\frac{1}{10}+\frac{1}{11}=$ ________



b) Which is closer to 1, 12/13 or 13/12 ? _______

c) Think: "What numbers should I add mentally to find this small distance?"___, ___

18) $\frac{100}{99}-\frac{99}{100}=\frac{1}{-}+\frac{1}{-}=$

Fill in

No quiz for Unit 35

QUIZ: THE ABYSS

Unit 36

1) Put the *new* 5-digit number in the bottom row at right by using George Cantor's diagonalization procedure, adding 1 each time.

3	8	6	**7**	**7**
2	4	8	7	8
1	8	5	6	9
1	7	8	7	6
6	1	3	4	4

2) Circle the letter of each that is infinite:

a) The negative integers

b) All numbers evenly divisible by 5

c) $\aleph_0$

d) $\aleph_1$

3) Use an exponent and express the cardinal number of the real numbers. _____

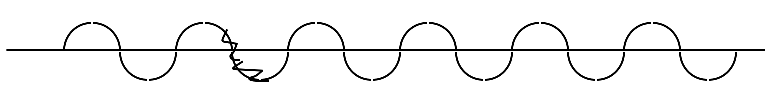
4) T/F: The rational numbers are a proper subset of the algebraic numbers *and* the set of *algebraic* numbers are all of the numbers which are solutions to algebraic equations. ___

5) Name two transcendental numbers that you know. _____ _____

Important Note: Because of the very advanced nature of some of the questions 6 – 18, feel free to go back to the unit and examine appropriate parts of Unit 36's pages 6 – 18. But be sure to try all the questions first.

6) Because the real numbers are uncountable and the algebraic numbers are countable, then the __________________ numbers account for the vast difference between the cardinal numbers of the countables and uncountables.

7)



If you were a centipede running back and forth across a straight line like this, what are the chances in such a trip of 500 yards that you would cross a point corresponding to a rational number? **a)** Very slim **b)** about 50/50 **c)** Very good **d)** Almost certain

QUIZ: THE ABYSS Unit 36

8) (Yes/ No): Is it possible to increase the size of a finite number by some calculations like finite exponents being themselves raised to finite powers or any other enlargement with finite numbers and operations? _____

9) T/F: Cantor had other mathematicians working with him to develop the Theory of Transfinites ___

10) T/F: An indirect proof states an assumption as true and then shows it to be false by showing a false consequence of the assumption. ____

11) T/F: Cantor used, among other things, an indirect proof in showing the decimals uncountable. ___

12) State your version of a hypothesis (not the Continuum Hypothesis) to be disproved in order to establish the non-countability of the reals. (Hint: This concerns a supposed list with a certain quality.)

____

____

13) Describe Cantor's diagonalization process in enough detail to show that you understand the process. Include what it can produce repeatedly.

____

____

14) Tell as precisely as you can what Cantor showed in order to establish that the set of real numbers is not countable. Your answer should include the ideas in ex. 12 and 13, and include exactly what he showed about any list of numbers that someone claimed to be countable.

____

____

QUIZ: THE ABYSS Unit 36

15) T/F: Cantor proved that the cardinal number of the power set of the natural numbers, $2^{\aleph_0}$, is also the cardinal number of the real numbers. ____

16) Consider the infinite string of infinite cardinal numbers: $\aleph_0$, $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, and on and on.

a) Write the next entry in the above string of cardinal numbers. ________________

b) Did Cantor think that sets having these cardinal numbers actually existed? ____

c) Did he prove it? _____

17) There was a giant fly in the ointment of the infinite string of infinities which Cantor formed by repeatedly applying the power set idea. This apparently fertile field for scholarly study would attract scholars only if this string started with the natural numbers as the first infinity and, as one might expect, progressed to the reals as the second infinity. Of course that is exactly what happens if we attend only to forming power sets and think that this will uncover *all* mysterious further sets of numbers.

a) Was Cantor aware of all this? _____.

b) Did the idea seem to bother him much at first? _____

c) In the end, what was he able to prove about it? ____________

d) The expressions

"The cardinal number of the natural numbers is $\aleph_0$, $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$ . . ."

are called collectively the ________________ ______________________ .

18) Insert proved/disproved, one or the other in each blank: In the 1930's Kurt Godel proved that the Continuum Hypothesis could not be a) _____________ using customary basic rules of set theory and in 1963 Paul Cohen proved that The Continuum Hypothesis could not be b) ___________ using those rules for sets.

The Internet reference <Peter Suber/Infinite Sets> is given again here because he writes very clearly about these ideas and more.

No quiz for Unit 37

QUIZ: RUSSELL'S PARADOX

Unit 38

Bertrand Russell showed that a paradox results when certain assignments are made with sets. He said that there are sets which are members of themselves and sets which are not members of themselves.

He then considered the set of sets which *are* members of themselves (call this set S) and the set of sets which *are not* members of themselves (call this set X).

You take it from here and tell what he did and what the results show.

QUIZ: TO END OR NOT TO END? IS THAT THE QUESTIONS? Unit 39

1a) In A below, complete the harmonic series as far as sixteen terms. There are first 2 fractions; then, a grouped pair of fractions. Starting with 1/5, complete a group of the next 3 decreasing fractions, then another group of 8 decreasing fractions and finally an indication of continuation.

b) Directly under the terms of the harmonic series, complete powers of 2 in the denominators as shown. There will be 4 one-eighths and then more of the next group.

c) In C**,** continue the row of one-halves, showing that each 1/2 represents the group above it.

A $\quad \frac{1}{1}+\frac{1}{2}+\left\{\frac{1}{3}+\frac{1}{4}\right\}+\left\{\frac{1}{5}+\right.$

B $\quad \frac{1}{1}+\frac{1}{2}+\left\{\frac{1}{4}+\frac{1}{4}\right\}+\left\{\frac{1}{8}+\right.$

C $\quad \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} \qquad + \frac{1}{2} + \;..$

*2a) .25 + .5 + .75 + 1 + _____ + _____ + _______ . . convergent or divergent? __________

b) 1/2 + 1 + 1 1/2 + _______ + _______. . . convergent or divergent? _____________

c) .1 + .01 + .001 + . . . + convergent or divergent? ____________ (Recall Achilles)

d) What is the sum of an infinite number of the terms in 2c) ? ______

3) T/F Each individual term of B is equal to or less than its corresponding term in A. ___

4) T/F The series in B and C are the same in value; they are merely written differently, *and*, C is divergent. ___

5) T/F C, having the same value as B, grows slower/faster than A ___

6) Your conclusion about the harmonic series: _______________________________________

____

QUIZ: RUSSIAN PEASANT MULTIPLICATION

Unit 40

Use Russian Peasant Multiplication:

1) 34 x 28
17 x 56
etc.

2) 17 x 13

3) 400 x 48

4) 32 x 102

Do these mentally (not by Russian Peasant method).

5) 14 x 16 = **6)** 13 x 17 =

7) 32 x 28 = **8)** 34 x 28 =

9) 30 x 100 = **10)** 32 x 100 =

11) 30 x 102= ***12)** 32 x 102=

13) 60 x 70 = **14)** 62 x 70 =

***15)** 62 x 72 = ***16)** 63 x 70=

17) In ex. 1 you treated the 17 as if it were _______.

18) In ex. 3 you treated the ______ as if it were ______.

***19)** We always add the bottom number because we always treat the 1 as if it were _____.

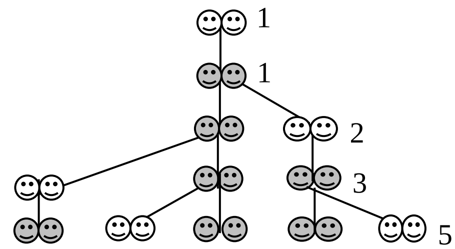
QUIZ: FIBONACCI AND FRIENDS

Unit 41

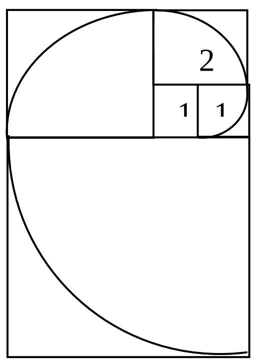
1) Fill the blanks for the early parts of the Fibonacci series: 1 ___ ___ ___ 5 ___ ___ ___ __

2) Add one more row to this diagram of "rabbits".

Shade only those pairs "old enough" to be parents.



3)



In the diagram above, draw the next larger square and its arc and put the next three numbers where they belong in the diagram.

4) Fill in the missing numbers in this Pascal's triangle.

1

1 1

1 2 1

1 3 3 1

1 ___ ___ ___ 1

1 5 10 10 5 1

1 6 ___ ___ ___ 6 1

1 7 ___ ___ ___ ___ 7 1

Work from the bottom up.

5a) $\dfrac{1}{1+\dfrac{1}{1}} =$ ____

b) $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1}}} =$ ____

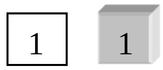
***c)** $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1}}}} =$ ____

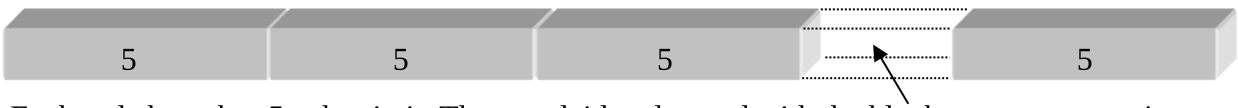
***d)** $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1}}}}} =$ ____

***e)** Look over a) through d) and construct above or on back the next logical exercise after d and solve it. You will probably use previous answer (recursion) in both the construction and solution.

***6)** Without any construction, write only the expected answers to **5g)** _____ **5h)** _____ **5i)** ____

QUIZ: RODS AND STAMPS — Unit 42





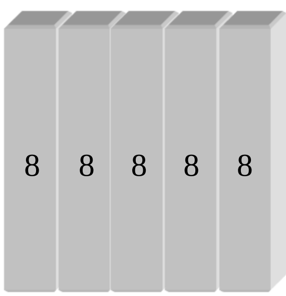
1) Each rod above has 5 cubes in it. They are laid end to end with the blank space representing many missing rods. The front, top, back and bottom are all free for stamping, and also the two ends.

a) How many stamps will cover 160 such 5-rods? _____

b) How many stamps for 6000 such 5-rods _______?

c) How many stamps for 1000 such 20-rods? ________

2)



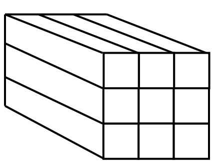
Any rod surface contacting another rod surface is unavailable for stamping.

a) How many stamps will cover all these available surfaces? ______

b) How many stamps for 5000 such 8-rods? ______

c) How many stamps for 10^4 such 8-rods? _________

3)

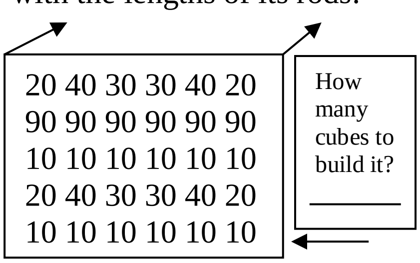


a) How many stamps to cover all outer surfaces including both ends, sides and top and bottom, if each rod is 1 stamp on its end and 7 cubes long?____

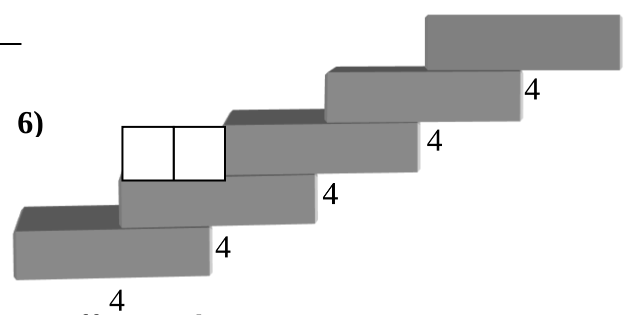
b) How many cubes to build another just like it? _____

4) How many cubes are needed to build a structure like exercise 3 but with its end 40 cubes wide and 50 cubes high and 600 cubes long? _________

***5)** The end of a stack is stamped with the lengths of its rods:



6)



The 4-rods above are offset 2 cubes.

6a) How many cubes are needed to build the structure?____

***6b)** How many stamps are needed to cover it, including the bottom, but not including the touching areas between the rods? ________

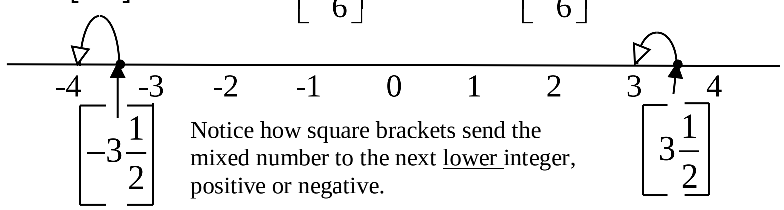
****6c)** How many stamps to cover a stack of 300 like these?

QUIZ: SQUARE BRACKETS 1 **Unit 43

<u>No Calculator</u>

1) $\left\lfloor 2\frac{3}{4} \right\rfloor =$ ____ **2)** $[19.5] =$ ____ **3)** $\left\lfloor \frac{11}{4} \right\rfloor =$ ____ **4)** $[180.3946] =$ ______

5) $[.3] =$ ___ **6)** $[7.9] =$ ____ **7)** $\left\lfloor 5\frac{5}{6} \right\rfloor =$ ____ **8)** $\left\lfloor 5\frac{7}{6} \right\rfloor =$ ____



9) $\left\lfloor -2\frac{1}{3} \right\rfloor =$ (Be careful) ____ **10)** $[8] =$ ___ **11)** $[-8] =$ ___ **12)** $[-8.2] =$ ____

13) $[-.01] =$ ____ **14)** The greatest integer in 3 ½ is ___. **15)** The greatest integer in -3 ½ is ____

16) $\left\lfloor \frac{1}{1000} \right\rfloor =$ ____ **17)** $\left\lfloor -\frac{1}{1000} \right\rfloor =$ ___ **18)** $\left\lfloor \frac{12\ 1/2}{11} \right\rfloor =$ ____ **19)** $\left\lfloor \frac{10}{4.91} \right\rfloor =$ ____

20) $\left\lfloor \frac{10}{5.001} \right\rfloor =$ ___ **21)** $[2.999] =$ ____ **22)** $\left\lfloor 2.\overline{9} \right\rfloor =$ ___ **23)** $\left\lfloor \frac{15.98}{4.1} \right\rfloor = \left[\quad\quad\right]$

24) $\left\lfloor \frac{125.15}{12.1} \right\rfloor =$ ____ **25)** $\left\lfloor \frac{120.15}{12.1} \right\rfloor =$ ____ **26)** $\left\lfloor 10^{-500} \right\rfloor =$ ____ **27)** $\left\lfloor \frac{7}{13} \right\rfloor + \left\lfloor \frac{51}{100} \right\rfloor =$ ____

28) $\left[\frac{7}{13} + \frac{51}{100}\right] =$ ____ **29)** $\frac{1}{2} + \left[\frac{1.02}{2.03}\right] =$ ____ **30)** $\left[\frac{7}{8} + .12\right] =$ ____ **31)** $\left\lfloor -\frac{7}{9} \right\rfloor =$ ____

32) $\left\lfloor \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right\rfloor =$ ____ **33)** $\left\lfloor \frac{1}{20} + \frac{1}{30} + \frac{1}{40} \right\rfloor =$ ____ **34)** $25 \times \left[\frac{7}{9}\right]^{100} =$ ____

QUIZ: SQUARE BRACKETS 2

****Unit 44**

1) $\left\lfloor \frac{1}{30} \times 60 \right\rfloor$ = ____ **2)** $\frac{1}{20} \times [61]$ = ____ **3)** $\left\lfloor \frac{1}{30} \times 59 \right\rfloor$ = ____ **4)** $\left\lfloor \frac{3}{25} \times 77 \right\rfloor$ = ____

5) $\left\lfloor \frac{29}{30} + \frac{1}{20} \right\rfloor$ = ____ **6)** $\left\lfloor \frac{29}{30} + \frac{1}{31} \right\rfloor$ = ____ **7)** $\left\lfloor \frac{142}{143} + \frac{1}{200} \right\rfloor$ = ____ **8)** $\left\lfloor 2\frac{1}{59} \times 60 \right\rfloor$ = ____

9) $\left\lfloor \frac{31}{30} + \frac{30}{31} \right\rfloor$ = ____ **10)** $\left\lfloor \frac{2}{3} + \frac{3}{2} \right\rfloor$ = ____ **11)** $\left\lfloor \frac{31}{40} + \frac{40}{31} \right\rfloor$ = ____ **12)** $\left\lfloor \frac{29}{28} + \frac{30}{29} \right\rfloor$ = ____

13) $\left\lfloor \frac{15}{17} + \frac{19}{17} \right\rfloor$ = ____ **14)** $\left[(.99)^2 \right]$ = ____ **15)** $\left[(.01)^{1000} \right]$ = ____ **16)** $\left[1.0\overline{3} \right]^{1000}$ = ____

17) $\left\lfloor \frac{8}{31} + \frac{8}{32} + \frac{8}{33} + \frac{8}{34} \right\rfloor$ = ____ **18)** $\left[20.\overline{9} - 20 \right]$ = ____ **19)** $\left\lfloor \frac{11}{33} - \frac{97}{99} \right\rfloor$ = ____

20) $\left\lfloor \frac{1}{305} - \frac{1}{304} \right\rfloor$ = ____ **21)** $\left[\sqrt{144} \right]$ = ____ **22)** $\left[\sqrt{1.44} \right]$ = ____ **23)** $\left[\dfrac{\frac{2}{3} \times 183^3}{\frac{3}{4} \times 183^3} \right]$ = ____

24) $\left\lfloor \sqrt{80} - 2\frac{1}{2} \right\rfloor$ = ____ **25)** $\left[\sqrt{\frac{50}{7}} \right]$ = ____ **26)** $\left\lfloor \frac{5}{6} + \frac{58}{59} \right\rfloor$ = ____ **27)** $\left\lfloor \frac{6}{\sqrt{35}} \right\rfloor$ = ____

28) $\dfrac{6}{\left[\sqrt{35} \right]}$ = ____ **29)** $\left[\sqrt{\sqrt{\sqrt{\sqrt{2.05}}}} \right]$ = ____ **30)** $\left[\sqrt{\sqrt{\sqrt{\sqrt{.93}}}} \right]$ = ____

QUIZ: SIMULTANEOUS EQUATIONS — Unit 45

1) $\bigcirc \times \square = 24$
$\bigcirc + \square = 10$

2) $\bigcirc + \square = 14$
$\bigcirc \times \square = 13$

3) $\bigcirc \times \square = 36$
$\bigcirc - \square = 0$

4) $\bigcirc + \square = 14$
$\bigcirc \times \square = 45$

5) $\bigcirc + \bigcirc + \square = 19$
$\bigcirc \times \square = 24$

6) $\square \times \triangle = 108$
$\square - \triangle = 52$

7) $\bigcirc \times \triangle = 121$
$3 \times \bigcirc - \triangle = 22$

8) $\bigcirc \times \square = 72$
$\bigcirc \div \square = 8$

9) $\text{⯄}^2 + \triangle^2 = 221$
$\text{⯄} - \triangle = 1$

10) $\bigcirc \times \square = 224$
$\bigcirc - \square = 2$

11) $\bigcirc^2 + \square^2 = 100$
$\bigcirc - \square = 2$

12) $\square^2 + \bigcirc^2 = 17$
$\square \div \bigcirc = 4$

13) $\square \div \sqrt{5} = \sqrt{5}$
$\square + \bigcirc = 18$

14) $\square + \bigcirc = 0$
$\square \times \bigcirc = -1$

***15)** $\square - \bigcirc = 9$
$\square + \bigcirc = 3$

16) $\square - \bigcirc = 6$
$\square \times \bigcirc = 91$

***17)** $\bigcirc \times \triangle = 5$
$\bigcirc - \triangle = 0$

18) $\square \times \bigcirc = 48$
$\square - \bigcirc = 13$

19) $\square \times \bigcirc = 32$
$\square \div \bigcirc = 2$

20) $\square \times \bigcirc = 1$
$\square + \bigcirc = 2\frac{1}{2}$

QUIZ: SQUARE BRACKETS 3 **Unit 46

For exercises 1 – 8: Circle each number which when put in the box, will make the statement true.

1) $\left[\square\right] = 6$ $\left\{6,\ 7,\ 6.5,\ 5.999\ldots,\ 3.9999,\ 2\frac{7}{2}\right\}$

2) $\left[\square\right] = 7$ $\left\{7\frac{6}{7},\ 7\frac{8}{7},\ 750\%,\ 7.\overline{9},\ 6.5,\ 6.\overline{9}\right\}$

3) $\left[\square + 8\right] = 13$ $\left\{4,\ 9,\ 9.\overline{9},\ 4.5,\ 5.\overline{9},\ 5.0001,\ 5.9999\right\}$

4) $\left[\square\right] = -4$ $\left\{-3.9,\ -4,\ -4.1,\ 4.1,\ -3.\overline{9},\ -3.1,\ -3\right\}$

5) $\left[\square\right] \times 7 = 14$ $\left\{2,\ 3,\ 1,\ 2\frac{1}{7},\ 2\frac{1}{4},\ 2\frac{1}{6}\right\}$

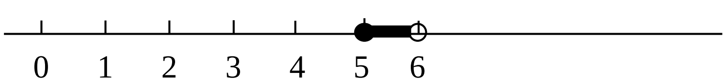
6) $6 + \left[\square\right] = -2$ $\left\{-2,\ 2,\ 2\frac{1}{2},\ 6,\ -6,\ -8,\ -8\frac{1}{2},\ -7\frac{1}{2},\ -7,\ -2\frac{1}{2},\ -7.25\right\}$

7) $\left[\square \div 5\right] = 4$ $\left\{21.5,\ 24.5,\ 25.5,\ 27.5,\ 10,\ 15\frac{1}{2},\ 20,\ 20.3\right\}$

8) $\left[\square + \frac{1}{4}\right] = \square + \frac{1}{4}$ $\left\{\frac{3}{4},\ 1,\ 1\frac{1}{4},\ -1,\ -\frac{3}{4},\ -1\frac{3}{4},\ -\frac{1}{4},\ 1\frac{3}{4},\ -5\frac{1}{4},\ 5\frac{3}{4}\right\}$

Reminder: Using the number line lets us show *all* the real numbers which make a statement true:

Sample : $\left[\square\right] = 5$



Insert numbers as needed for graphing the following:

9) $\left[\square\right] = -8$

10) $5 + \left[\square\right] = -2$

11) $\left[\frac{\square}{5}\right] = 8$

12) $\left[\frac{\square + 3}{6}\right] = 2$

13) Show at least two negatives and two positives on this the graph: $\square + \frac{1}{3} = \left[\square + \frac{1}{3}\right]$

0

14) $\left[\frac{\square}{3}\right] = \left[\frac{\square}{4}\right]$

0

15) $\left[\frac{\square}{2}\right] = \left[\frac{\square}{3}\right]$

0

QUIZ: RAMANUJAN

Unit 47

1) Write the first eight members of the triangular number series.

2) Write the first eight members of the Ramanujan series.(Fractions)

3) Write the first eight members of the Halfwalk series.

4) Write the first eight members of the harmonic Series.

Two of the “changes” in the unit *Ramanujan* resulted in

$$2 \text{ x } \left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} \qquad \ldots \right)$$

Regrouping, we see:

See exactly what each bracket encloses and how this affects the total.

5) Show the regrouping that we saw.

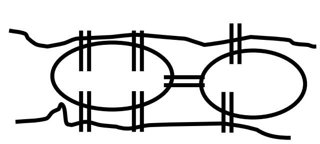
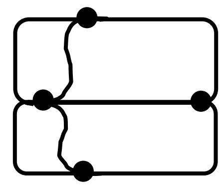
6) One easy step from here and we know that Ramanujan showed his series to be convergent/divergent __________, and the exact value for the sum of the Ramanujan series is ___ .

7) Write the series of doubles of the triangular numbers in exercise 1.

What characteristic of each of these triangular number doubles enabled the changes which came afterward?

QUIZ: BRIDGES OF KONIGSBERG

Unit 48

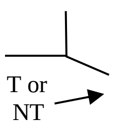



Reminder: To trace, or is traceable, is understood to mean that a complete tracing can be made without any retracing or jumping on or off the diagram.

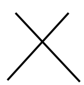
To prove that a thing is possible often means simply finding a way to do it. But failure to find a way does not necessarily mean that it cannot be done. Proving a thing impossible can be difficult.

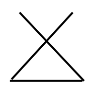
1) Leonard __________ (pronounced "oiler") made a thorough analysis of the traceability of points and lines which correspond with land masses and bridges, respectively. This transformation of the problem led to a whole new branch of mathematics called *Topology*.

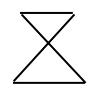
2) Use T to indicate that a diagram is traceable, NT for not traceable:

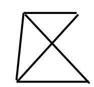
a. 

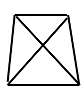
___

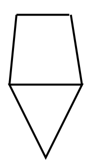
b.  ___

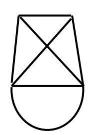
c.  ___

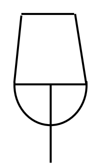
d.  ___

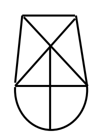
e.  ___

f.  ___

g.  ___

h.  ___

i.  ___

j.  ___

3) T/F A traceable figure :

a. May have 2 even vertices.___

b. May have 2 odd vertices.___

c. May have any number of even vertices. ___

d. May have any number of odd vertices. ___

e. Must have at least 1 odd vertex ___

f. May have no odd vertices. ___

g. Must have at least 2 even vertices. ___

h. Must have at least 2 odd vertices. ___

i. With 2 odd vertices, must have the tracing start at one of them and end at the other. ___

j. With all even vertices, may have the tracing start at any vertex ___

4) State the exact requirements regarding odd or even vertices for a figure to be traceable.

____

____

QUIZ: LOGIC AND SYMBOLS

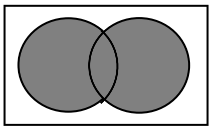
***Unit 49

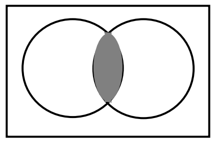
1) Put the letter of each object related to "or" in the "or" row below, and the letter of each object related to "and" in the "and" row below.

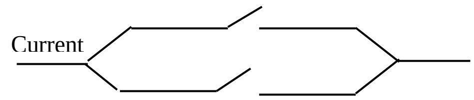
"or":_______________________________

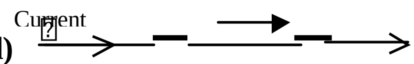
"and": ________________________________

Special order not required.

a) 

b) 

c) 

d) 

e) disjunction

f) conjunction

g) Input Outputs

a	b	
0	0	0
0	1	0
1	0	0
1	1	1

h) Input Outputs

a	b	
0	0	0
0	1	1
1	0	1
1	1	1

i) $\vee$

j) $\wedge$

2) ~a means __________; ~ ~ a $\leftrightarrow$ ___

3) Write in symbols: "The not of a <u>and</u> the not of b"

4) Write in symbols: "The not of a <u>or</u> b"

5) Complete the truth table below that you did in Unit 49.

1	2	3	4	5	6	7	8	9	10
a	**b**	**~a**	**~b**	**a $\wedge$ b**	**a $\vee$ b**	**~(a $\wedge$ b)**	**~(a $\vee$ b)**	**~a $\vee$ ~b**	**~a $\wedge$ ~b**
0	**0**	**1**	**1**	**0**	**0**				
				0	**1**				

DeMorgan's law (1),

6a) Complete: The "not" of the conjunction of two statements is equivalent to the disjunction of ____.

b) Using **r** and **t** as statements, complete part **a)** in symbols: **~(r $\wedge$ t)$\leftrightarrow$**______________

c) T/F: ~(r $\vee$ t) $\leftrightarrow$ ~r $\wedge$ ~t ___ **d)** DeMorgan's law (2) says in words: The not of the disjunction of ____

7a) Given (r $\rightarrow$t), do we really know that t is true? ____ b) Does (r $\wedge$(r $\rightarrow$t)) give us t? _____

QUIZ: FLOW CHARTS — Unit 50

The form of the plan is changed in this quiz from a flow chart to a numbered list. This is a little closer to computer programming and can be used for programs which are not complex.

1) 1 Enter Goal ____ (Try 10) (The user designates where to end the running of the program)

2 Let Count = 1

3 Let Num = 2 (This gives Num its value. Print "Num" prints the *word* Num.

4 Print Count

5 Print Num Print Num prints its present value

6 Let Count = Count + 1 (Increases the value of Count by 1)

7 If Count $\geq$ Goal, then stop ($\geq$ means "is equal to, or greater than)

8 Let Num = Num + 2

Set a goal, say 10, and record the results

Count: ___ ___ ___ ___ ___ ___ ___ ___ ___ ___

Num: ___ ___ ___ ___ ___ ___ ___ ___ ___ ___

***2)** 1 Let Count = 1

2 Let Num = 1

3 Print Count

4 Print Num

5 Let Count = Count + 1

6 Let Num = Num + Count (You will see how this develops as you cycle through the program.)

7 If Count $\geq$ 11 then Stop
(If Count equals or exceeds 11 then stop)

Write each number in its space below as you cycle through the program loop.

Count: ___ ___ ___ ___ ___ ___ ___ ___ ___ ___

Num: ___ ___ ___ ___ ___ ___ ___ ___ ___ ___

***3)** Now it is your turn. Write a program that will print out all of the numbers divisible by 5 that start at 100 and end after printing the number 10,000. You will not need a Count. You might want to use >, which means "is greater than".

1 Let Goal = 10,000
2 Let Num = 100
3

QUIZ: PERMUTATIONS 1

Unit 51

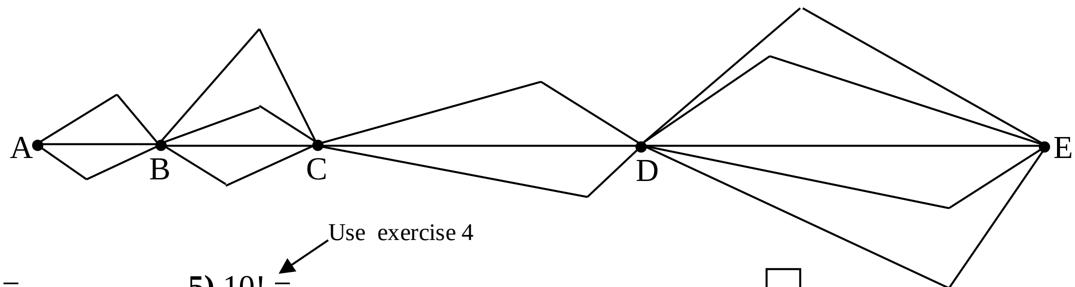
1) Complete the listing of all the permutations of the digits 1,2,3:

1 2 3   2 _ _   ____
1 3 2   ____   ____

***2)** Complete the list of permutations of a, b, c, d.

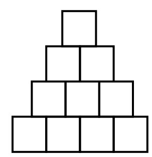
abcd   bacd   ____   ____
abdc
acbd
____
____
____

3) In how many ways can one travel from point A to point E ?______



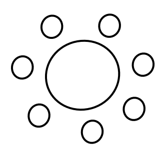
Use exercise 4

4) 9! = ________   **5)** 10! = ______



6) In how many ways may the digits 0 - 9 be arranged in these boxes, no repetitions? _______

7) In how many ways may the seven dwarfs be arranged in a circle around Snow White? _______



Consider the same order with a different starting point as the same.

8) Find the value of the expression n x (n – 1) x (n – 2) x . . . x 3 x 2 x 1,

a) When n = 10 ______

b) When n = 11 ______

Most of the remaining examples can be none mentally, perhaps not 11**d)**

9) 13! = 12! x ____.   **10a)** $12! \div 10! =$ ____   b) $\frac{1000!}{999!} =$ ______   **c)** $\frac{9!}{10!} =$ ______

11a) $\frac{10!}{8! \times 2!} =$ _______   **b)** $\frac{8! \times 2!}{10!} =$ ______   **c) T/F** 8! x 2! = 10! ___   **d)** $\frac{6! \times 2!}{5! \times 4!} =$ _______

QUIZ: PERMUTATIONS 2 AND TOUGHIES

Unit 52 – 53

1) How many ways can the symbols +, x, -, 0 be arranged? ______ (Don't include commas)

2) How many arrangements can be made of 5 objects:

a) Taken 3 at a time? _____

b) Taken 2 at a time? _____

3) Using 1, 2, 3, 4, 5, 6, how many arrangements can be made:

a) With all of the digits? ____

b) Taken 5 at a time? ____ (Surprised?)

c) Taken 3 at a time? ____

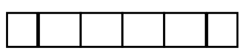
4a) How many permutations can be made of 50 objects? ____ (Give answer in the *form* of __!)

b) How many arrangements can be made of 50 objects taken 2 at a time? _____

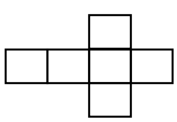
c) How many arrangements can be made of 50 objects taken 2 at a time and allowing objects to be repeated ? ________

5a) How many permutation of e, f, g, h may be made taken three at a time? ____

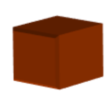
***b)** List all of the three-member permutations of Exercise 5a). Use a system and help yourself by first finding out how many permutations to expect (check 5a). *do work here*

****6)** How many public car registration number plates can be made having any numeral on each end except 0 or 1, with repetitions; any letter in each of the 4 middle places except O (Oh) and l (el), with repetitions. 

7) Suppose you are to paint a cube with each side a different color (given 6 colors) and with no two cubes painted in the same pattern. Exactly how many unpainted cubes do you need? ____



(Cube unfolded)



QUIZ: SOME PECULIAR THINGS ABOUT ZERO — Unit 54

1a) If $8 \div 2 = 4$, then 4 x ___ = 8 **b)** If $8 \div 0 = 8$, then 8 x 0 = ___ **c)** If $9 \div 1 = 9$, then 9 x 1 = ___

*d) If $9 \div 1 = 1$, then 1 x ___ = 9 **e)** Which of a) to d) give(s) a false "if"? ________

2a) If $7 \div 2\frac{1}{3} = 3$, (Does it? ____) , then 3 x ___ = 7 . Does it? _____

***b)** If 0 ÷ 8 = 0 (Does it? ____), then ___ x ___ = ___

c) If 0 ÷ $2.89\frac{1}{17}$ = 0 (Does it? ___), then ___ x ___ = ____

d) T/F If t = 0 or y = 0, then $\frac{t \times y}{5} = 0$ ____.

3a) If 0 ÷ 0 = 0, then ___ **x** ___ = 0. Does it? _____ **b)** If 0 ÷ 0 = 57 ½, then ___ x ___ = ____ . Does it? ____ Explain: ____

	Make Impossible	Make Indeterminate	Make Zero
Sample 1 * $\frac{10 \div y}{10 - z}$	y ≠ 0, z = 10	y = 0 and z = 10	No values will work
Sample 2 $\frac{t}{r + 0.5}$	t ≠ 0, r = -0.5	t = 0, r = -0.5	t = 0, r ≠ -0.5
4) $\frac{a - b}{c}$			
5) $\frac{37\ 1/2\% - a}{b}$			
6) $\frac{a}{b} \times 5$			
7) $\frac{1}{\frac{a}{b}}$			

QUIZ: COMBINATIONS AND TOUGHIES

Unit 55 – ***56

1) From a combination of 3 letters abc,

(a) Write out the permutations taken 2 at a time.

(b) Write out the combinations that can be made taken 2 at a time.

a)

b)

2) From a combination of 8 letters,

a) How many permutations of all 8 letters may be made? ______

b) How many permutations having 3 of the 8 original letters may made? _____

*****c) How many 3 letter combinations may be made from the original 8 letters? ______

3) Suppose we have 14 hockey players, 10 forwards and 4 defensemen. How many ways could you combine 3 fowards and 2 defensemen to make complete teams of 5? ______________

Evaluate: (Show work on this paper)

4) ${}_6C_3 =$

5) ${}_{30}C_5 =$

6) ${}_6C_4 =$

7) $\frac{{}_8C_7}{{}_7C_6} =$

8) ${}_{1000}C_{999} =$

9) ${}_{102}C\_\_\_\_ = 102$

?

***10)** ${}_{2015}C_{2015} =$ ______

QUIZ: MODULAR ARITHMETIC *Unit 57

1) Complete the addition table Mod 6.

Recall that 6 does not appear; use 0.

2) 4 + 7 ≡___ mod 6

3) 5 + 3 ≡ ___ mod 6 = read "5 + 3 is congruent to ___ mod 6." (Fill blanks).

4) 5 + 5 + 5 + 5 ≡___ mod 6.

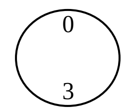
5) Complete multiplication table mod 6.

+	0	1	2	3	4	5
0	0	1				
1		2				
2				5		
3						
4						
5						

Mod 6

x	0	1	2	3	4	5
0	0	0				
1		1				
2				0		
3						
4						
5						

If more than one number works, give the least whole number possible



6) 6 x 6≡___ mod 6

7) 6 x 6 x 6 ≡ ___ mod 9

***8)** 6^{150} ≡___ mod 6

***9)** 6^{1000} ≡___ mod 6

10) 6 x 7 ≡___ mod 6

11) 6 x 7 ≡ ___ mod 7

12) 6 x 9 ≡___ mod 7

13) 7 x ___ ≡ 2 mod 6

14) 9^2 ≡___ mod 8

15) 4^2 ≡ ___ mod 5

16) 1 + 4 ≡___ mod 7

17) 3 – 5 ≡___mod 6

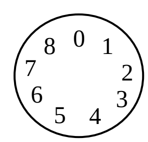
18) 1 – 2 ≡___ mod 6

***19)** 2 – 3 ≡ 7 mod ___

***20)** 2 + 3 ≡3 mod ___

21) 5 x 7 ≡2 mod ___

mod 9



QUIZ: MODULAR ARITHMETIC 2 **Unit 58

In these, more than one answer is possible. Give all.

1) The two numbers 27 and 32:

a) Have a difference of _______

b) When divided by _____, have the same remainder

c) Can be said to be c o n _ _ _ _ _ _ with respect to ________.

d) Or, even better, $27 \equiv 32 \bmod$ ____

2a) The 7 jumper that starts on 3 on the mod 7 clock lands first on______.

b) The numbers 472 and 489 would have the same value on the mod ____ clock.

c) 630 and 635 are congruent with respect to mod _____ (1 is an answer, but we will ignore that.)

d) 2801 and 2815 are congruent with respect to what number greater than 10? ____

e) $2800 \equiv 2815 \bmod$ _____ (Give two answers).

3) Give all natural numbers such that:

Sample: $\square \equiv 3 \bmod 8$. Ans: 11, 19, 27, 35 . . .

a) ??? $\equiv$ 4 mod 13, ________________________

b) ??? $\equiv$ 12 mod 2. ________________________

c) ??? $\equiv$ 101 mod 13 ________________________

N means not necessarily; could be.

4) T/F/N. If $\frac{p-q}{t}$ = any natural number, then

a) T/F/N p and q are both divisible by t.______

b) T/F/N $p \equiv q \bmod t$. ______

c) T/F/N $p \div t$ and $q \div t$ have the same remainder ______

QUIZ: CARD FLIPS, GROUPS AND FIELDS

Unit 59

Mod 4

+	0	1	2	3
0				
1				
2				
3				

Card Flips

○	I	V	H	R
I				
V				
H				
R				

1) Completely fill in both tables above.

2)There are several interesting patterns in the above tables. By inspecting the completed table patterns:

a) Name the *identity* element for Mod 4 ____, and for Card Flips ____.

Recall that combining an element with its inverse gives the identity element:

b) Name the *inverse* for 1 in Mod 4 ____, and for V in Card Flips ____.

*__c)__ Describe the characteristic of either pattern that tells you that it is *abelian*.

____

____

3) State the *associative property* using symbols from the Flip Card table.

____

4) Name the four properties required for a Group and the fifth property often present.

Hint: C, I, I, A, (C)

__________ __________ __________ __________ __________

5) To move on from Groups to Fields, we need a second group and a property which "connects" them. Give the name of the property and show how it works.

Name of property ________________ . Example which displays it. ____________________

QUIZ: PRIMES 1 | Unit 60

No Calculator

1) For what numbers is this a rule? **"**If the sum of the digits of a number is divisible by a number, then so is the number divisible by it". ___,___ (Two 1 digit numbers)

2) T/F. If a number is not divisible by 3, then it is not divisible by any multiple of 3. ___

3) Name the lowest 2 digit prime ending in 1 ___; name the highest ___.

4) Is 911 divisible by <u>any</u> even number? ____

5) Is 911 divisible by <u>any</u> multiple of 3? ____

6a) How high must one test to see whether 911is prime or composite? _____

6b) Is 911 prime? _____

***7a)** To test whether 101 is prime, what is the <u>greatest</u> number you really need to divide by?___

b)Is 101 prime?____

(The rest of the numbers can be eliminated by rules of divisibility, multiples and $\sqrt{101}$.)

8) T/F To test a number as prime, use rules for divisibility, their multiples, and then divide by primes, stopping when you reach the square root (approximately) of the number being tested..

____

Sample: The prime factorization of 2400 = 24 x 100 = 2^3 x 3 x 10^2 = 2^3 x 3 x 2^2 x 5^2 = 2^5 x 3 x 5^2.

9) Give the prime factorizations of :

a) 1260 = ____

b) 990 = ____

10) T/F. a) 2 is a factor of 22 + 45 ____ **b)** of 22 + 46 ____ **c)** of 423 + 547 ____

11) T/F a) a^3 is a factor of a^5. ___ **b)** a^2b^3 is a factor of a^4b ___. **c)** a^2b^4 is a factor of a^2b^4 ___.

12) Discover without a calculator whether: **T/F** a) 2^4 x 3^2 x 5^4 is a factor of 30000 _____

b) 2 x 3^4 x 5 is a factor of 540 ______ **c)** 2^2 x 5^2 is a factor 1250 ______

13) Tell how many prime factors in each.

a) 2^3 x 3^2 ____ **b)** 3^3 x 5 x 7^2 ____ **c)** 2^{29} x 5 ____

d) 2^9 x 3^9 x 5^{99} x 7 ____ **e)** 2 x 3^3 x 5 x 7^{10} x 13^3 x 101^{39} ____

f) 2^5 x 3^3 x 17^3 x 19^5 ____

QUIZ: PRIME 2 — Unit 61

Handy reference:

a) If you multiply 2 or more of these odd numbers together, the answer will be *odd.*
$1 \times 3 \times 5 \times 7 \times 9 \times 11$

b) If you insert one or more evens into such a string of odds, the answer will be *even*

c) Even times even is *even* **d)** Odd times an odd is *odd* **e)** Odd times even is *even*

f) Consecutive natural numbers have no factors in common except 1; obvious but important.

Reminders: 42 + 49 = 7 x (6 + 7). (31! + 31) has 31 as a factor. 15! has 24 as a factor. (15! has 6 and 4 as factors, or 8 and 3, etc.)

T/F:

1a) 23! has 13 as a factor ___. → **b)** (25! + 3) has 3 as a factor ___.

***c)** (24! + 25) has 24 as a factor ___. **d)** 500**!** has (499 x 498) as a factor ___.

e) 17! has 8 x 9 as a factor ___. **f)** (17! + 1) has 8 x 9 as a factor ___.

g) 1001! has 631 as a factor ___. **h)** (1001! + 1) has 631 as a factor ___.

*** i)** 5160! has a factor of 10^3! ___ **j)** 51600! + 1 has a factor of 10^4!___.

k) 18! and 17! share more than 30 factors ___ **l)** 13 x 23 x 7 is a factor 100! + 1 ___.

m) 137 + 300! has 137 as a factor ___ **n)** 137 + 301! has 137 as a factor___.

o) 138 + 300! has 137 as a factor ___.

Correct to 1a-o). Understanding errors is especially important here.

2a) T/F Aside from 1, 31 is the only common factor of 37! and 29___.

b) T/F Aside from 1, 7 is the only common factor of 23! and 7 ___ .

c) T/F Aside from 1, 3 is the only common factor of 23! and 6 ___.

3a) T/F 636 is a common factor of 10,001! and 636.___. We can restate this as

636 is a factor of (10,001! + 636) ___.

b) T/F 637 is a factor of (10,001! + 637)___.

c) T/F 638 is a factor of (10,001! + 638)___

d) T/F In exercise **3)** your agreement has actually identified 3 very large numbers (each much larger than the number 1,000,000,000,000,000,000). ___

QUIZ: PRIME 2 — Unit 61

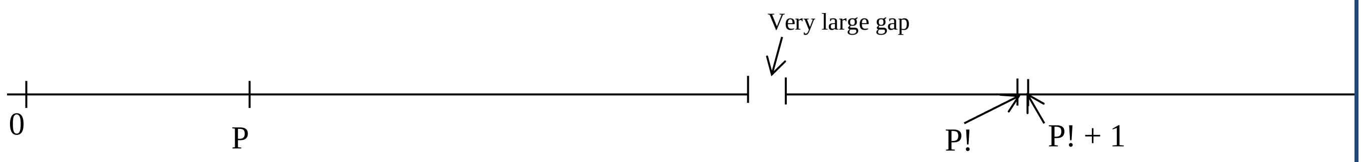
***4a)** Starting with 1001! + 2, write the next 3 consecutive composites.

"Next" does not include 20,000,001!

****b)** Starting with 20,000,001!, write an expression of the next 100 consecutive composites. Leave out most of them but indicate their presence and state the last number.

5a) List the prime factors of 30 ______ .

***b)** List the prime factors of 10 billion. **(**There are only a few.) ________________



6) Suppose it is claimed that P, a very large number, **is** the largest prime number. There are many larger numbers, but, it is claimed, none of them is prime.

Primes thin out as we climb the natural numbers so it seems that they might choke off somewhere.

For large numbers, Is P! prime? No, it ends in zero. Is P! + 1 prime? We don't know.

Is (P + 1)! prime**?** No, it ends in zero and has ***many*** factors

a) Do P! and P! + 1 share any factors apart from 1 and how do you know?___________________

If P is a large prime, then P! is very, very large, ends in 0, and so is not prime. But P! + 1 is a very large number which ends in 1. (Do you see why?) We do not know whether it is prime or not.

**** b)** Aside from 1, can P! + 1 have a factor smaller than P? Explain. _____________________

____

***c)** Therefore, If P! + 1 is not itself prime, then it has at least one prime factor larger than ________. T/F Thus it is proved that there is no last prime. ___

QUIZ ANSWERS

Units 1 and 2

1) 125 **2)** = 400 **3)** Twenty zeros **4)** 4096 **5)** 65,536

6) 28301.72 **7)** 12,312,300 **8)** 13, ^{5} **9)** 8/125

10a) 4/5 ***b)** ^{10}

11a) 1/5 **b)** 420 **c)** 100 **d)** 5

12) 8 ****13)** ? = 3

Units 3 - 4

1) right **2)** left, down **3a)** 2 **b)** 4 **4)** 7

5) 8 **6)** 1001, 9 1010, 10 1011, 11 1100, 12 **7)** 10

8a) 15 **b)** 11 **c)** 17 **d)** 10010

9a) 16 **b)** 64 ***c)** ^{8} **d)** ^{10} ***e)** 1

10) 11111 **11)** 63 **12)** 111111, 63 **13)** ^{5}, 31 **14)** 15

15) ^{33}**,** 8589934591 Calculator okay

Units 5 - 7

1) 43, 53, 63, 73 **2)** 43, 67 and 8019, 28019 and 28099 **3)** T **4)** 5

5) 108, 116, 124, 132 **6)** 22 **7)** -4, -8, -12, -16

8) -2, -9, -16, -23 **9)** -27, -22, -17, -12

10) -5609, -5617, -5625, -5633 **11)** -8, 4, 16, 28 **12)** Circle a,b,c,d,g,h

13a) 6 **b)** 12000 **c)** 10^{6} or 1,000,000 **d)** 21 x 10^{5} or 2,100,000

14a) 27,24, 21, 18 **b)** 26, 22, 18, 14 **c)** 25, 20, 15, 10

15) 60, 72, 84, 96, 108, 120, 132 **16)** 600, 660, 720, 780, 840, 900, 960

17) 4,21 , 9,29 **18)** 23 **19a)** F **b)** F **c)** F **d)** T **e)** F **20)** T **21)** Yes

22) 1, 2, 3, 2 **23)** Sometimes **24)** T **25)** 36

QUIZ ANSWERS

Unit 8

1) 72000	**2)** 63000	**3)** ^{4}	**4)** 54, ^{5}	**5)** 900,000
6) 1080	**7)** 36000	**8)** 1100	**9)** 12, ^{2}	**10)** 9, ^{7}
11) 2	**12)** 8	**13)** 3	**14)** 4	**15)** 108 x 10^{14}
16) 99,000,000 or 99 x 10^{6}		**17)** 12	**18)** 20000	**19)** 120,000
20) 72000	**21)** 600	**22)** 5000	**23)** 9	**24)** 45
25) 108				

26)
27
36
45
54
63
72
81
90
99
108

27)
108
117
126
135
144
153
162
171
180
189
198

28) 10^{9} is the only number that should ***not* be circled.**

Unit 9

1a) 100	**b)** 80	**c)** 80	**d)** 100
2a) -80	**b)** -100	**c)** -100	**d)** -80
3a) 53	**b)** 67	**c)** 67	**d)** 53
4a) 48	**b)** 48	**c)** 32	**d)** 32
5a) 19	**b)** 8000	**c)** 0	
6) 10,004	**7)** -10004	**8)** 2.5	**9)** 13.3
10) 3000	**11)** -6600	**12)** -2200	

QUIZ ANSWERS

Unit 10

1a) Associative **b)** (a x b) x c = a x (b x c) **2)** F **3)** ^{5} **4)** .37261 **5)** 20.56483 **6)** 20.56483

7) 2056000 **8)** .0002056 **9)** .0002056 **10)** 5689 **11)** .5689 **12)** $^{-1}$ **13)** ^{1} **14)** ^{0}

15a) 18 **b)** 18 **c)** .02134 **16)** 6 **17)** 1/5 **18)** 1 **19)** 7/3 or 2 1/3, 1/5

18 18 .02134

20) 0, identity **21)** Zero has no reciprocal. **22)** 1

23) The only false answers are c), h) and y).

There is no quiz for Unit 11

Unit 12

1) One to one correspondence. **2)** First **3)** Yes **4a)** T **b)** T **c)** F **d)** T **5)** T **6)** T **7a)** T **b)** 9900 **c)** 31 **d)** 401 **e)** A set which is one to one correspondence with a proper subset of itself.

Unit 13

1a) 2/5 **b)** 1/2 **c) I.** 5/2 or 2 ½ **II.** 2 **d)** 4 ½ **e)** reciprocal or opposite , 2/9 hours or 13 1/3 min (Final answer)

2) $\dfrac{1}{\frac{5}{2}+2} = \dfrac{1}{\frac{9}{2}} = \dfrac{2}{9}$ hrs.**;** 2/9 x 60min. = 13 1/3 min. **3)** $22\frac{2}{9}$ minutes

4) $\dfrac{1}{\frac{1}{\frac{1}{2}}+\frac{1}{\frac{1}{4}}+\frac{1}{\frac{1}{6}}} = \dfrac{1}{2+4+6} = \dfrac{1}{12}$ hours per trough = 5 minutes to fill a trough

Units 14 - 16

a. 7, 8, 9, 10 **b.** 49, 64, 81, 100 **c.** 56, 72, 90, 110 **d.** 28, 36, 45, 55 **1)** See **a.** above **2)** 2nd is 3 more, 3rd is 5 more; each growth is 2 larger. **3)** $n^2 + n$ is sum of number in **a** with number in **b**.

4) 18,003,000 **5)** 1,562,800

QUIZ ANSWERS

Unit 17

1) Every set **2)** null **3)** T **4a)** F **b)** F **c)** T

5) $\{\{t, u, v\}, \{t,u\}, \{t,v\}, \{u,v\}, \{t\}, \{u\}, \{v\}, \{\ \}$ or $\varnothing\}$

6) 8 **7)** 2^{11} or 2048 **8)** 2^{10} or 1024 **9)** 2^{250} ***10)** 8 because $2^8 = 256$

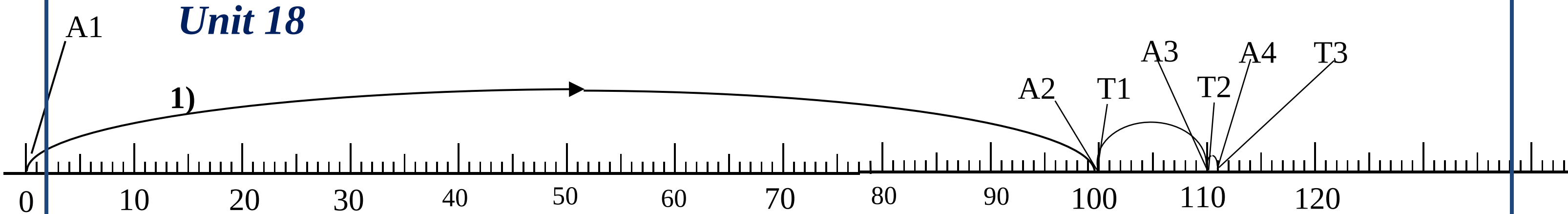
11a) F **b)** F **c)** T **12a)** F **b)** T **c)** T **d)** T **13a)** F **b)** T **c)** F **d)** F

14a) 1,3 **b)** $\{1,3\}, \{1\}, \{3\}, \varnothing$ or $\{\ \}$ **c)** $\varnothing$ **d)** 2, 5, $\{1,3\}$ **e)** 8

***f)** $\{\{2,5,\{1,3\}\}, \{2,5\}, \{2, \{1,3\}\}, \{5, \{1,3\}\}, \{2\}, \{5\}, \{\{1,3\}\}, \{\ \}\}$ or $\varnothing$

Unit 18

1)



2)

				Difference, T minus A
A1---	0	T1---	100	100
A2---	100	T2---	110	10
A3---	110	T3---	111	1
A4---	111	T4---	111.1	0.1
A5---	111.1	T5---	111.11	0.01
A6---	111.11	T5---	111.111	0.001
				0.0001
				0.00001
				0.000001

3) 100, 10^{-6} **4)** 11 1/9, or 11.1 1/9, etc.

Unit 19

1) T **2)** 1/16, 1/32, 1/64 **3)** 1/256, 1/1024 **4)** ^{100} **5a)** 1/524288 **b)** 1/536870912 **5c)** $1/2^{86}$

6) 15/16, 31/32, 63/64 **7)** ^{6}, 1 ; ^{11,} ^{11}; $\frac{2047}{2048}$ **8)** 1/32, 1/64, 1/128, 1/256, 1/512, 1/1024, 1/2048, 1/4096. **9)** 511/512, 1023/1024, 2047/2048, 4095/4096, 8191/8192 **10a)** 3 **b)** 4 **c)** 10

Units 20-21: No Quizzes

QUIZ ANSWERS

Unit 22

1) Yes **2)** 56 **3)** Yes **4)** 8, No **5)** 255 **6a)** 10000000 **b)** 11111111
7a) 3 **b)** 8 **c)** 33 **d)** 25443 **8) T/F** T **9)** 20,100 T **10)** 55,048

Unit 23

1) 1, 4; 1 is correct **2a)** 4 **b)** 8 **3a)** No **b)** Yes **c)** Yes **4)** 9 **5)** 6 **6)** b or 5 **7)** b a c
8) 4800 **9)** 270 **10)** 160,801

Unit 24

1) 5 **2)** 15 **3)** 7 **4)** 5.8 **5)** 7.1 **6a)** 20 **b)** 300 **c)** 120,000 ***d)** .95 or 1 **e)** .95 or .946
f) .03 **g)** 9 **h)** 8 **i)** 13.4 **j)** $\sqrt{6}$ **k)** 57 **l)** 7 **7)** 108 sq. in. ***b)** 17

Unit 25: No Quiz

Unit 26

1) 120 **2)** 60 **3)** 600 **4)** 60000 **5)** 500 **6)** 108 **7)** 1800 **8)** 900 **9)** 72 **10)** 14,400
11)1,200,000 **12)** 48 **13)** 1200 **14)** 144 **15)** 132 **16)** 1.32 **17)** 288 **18)** 960 **19)** 96 **20)** 96
21) 84 **22)** ^{3} **23)** 36 **24)** 48 **25)** 108 **26)** 216 **27)** 84 **28)** 168 **29)** 16,800 **30)** 1680 **31)**
1440 **32)** 8400 **33)** 4200 **34)** 144000 **35)** 120 **36)** 132 **37)** 132 **38)** 121 **39a)** 1221
39b) 12321 **40)** 6, 72 **41)** 4, 48 **42)** 143 **43)** 12 **44)** 144 **45)** 84 ***46)** 60 ***47)** 242 **48)**
484

Unit 27

1) 25 **2)** 12 ½ **3)** 75 **4)** 62 ½ **5)** 5/8 **6)** 62 ½ **7)** 12 ½ **8)** 10 **9)** 20 **10)** 60 **11)** 80
12) ½ **13)** 1/8 **14)** 62½% **15)** 62 ½ or 5/8 **16)** 6 ¼ , 1/16 **17)** 10 **18)** 40 **19)** 50
20) 9 **21)** ½ or .5 **22) .**33 or .3 or .33 1/3 **23)** .333… **24)** .33 1/3 **25) .**66 2/3 or $.66\overline{6}$ **26)** $.55\overline{5}$
27) 1/3 **28) .**333 . . . **29)** .666… **30)** 1 **31)** T **32)** 1/50 **33)** 3/100 **34)** .035 **35)** .0325
36) .0375 **37)** 1/7 **38)** 6/7 **39)** 2/35 **40)** 3/40 **41)** 80/3 or 26 2/3 **42)** F **43)** T

QUIZ ANSWERS

Unit 28

Note similarity
with 13^2 answer

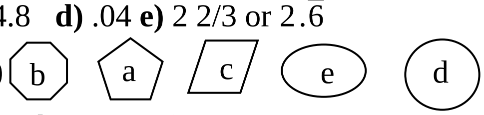
1) ^{2} **2)** ^{3} **3)** ^{6} **4)** 100 **5)** 121 **6)** 144 **7)** 156 **8)** 169 **9)** 196 **10)** 210 **11)** 225 **12)** 2.25 **13)** .0225 **14)** 2 5 **15)** 6.25 **16)** 2 ¼ **17)** 6 ¼ **18)** 12 ¼ **19)** 30 ¼ **20)** 20 ¼ **21)** 90 ¼ **22)** 56 ¼ **23)** 56.25 **24)** 5,625 **25)** 2025 **26)** 722,500 **27)** 65 **28)** 10, 11 **29)** 100 **30)** 100, 121 **31)** 400, 441, 484, 529 **32)** 1,000,000 1,002,001 1,004,004 **33)** 225, 256, 289, 324 **34a)** 228, 259, 292, 18, 22, 75 ¼, 3 ½, **34b)** 1805, 455, 517, 887, 4, 40 **35)** 100, 99, 96 **36)** 400, 399, 396 **37)** 899, 896, 891, 884 **38) a)** 41 x 39 **b)** 62 x 58 **c)** 73 x 67

Units 29 - 30

1) 320
2) 640
3) 12800
4) 80
5) 400
6) 10
7) 6 ¼
8) 24,680
9) 100
10) 800
11) 578
12) 721
13) 445
14) 41
15) 38 ½
16) 37 2/3
17) 55 ¾
18) 49
19) 98
20) 196
21) 196
22) 392
23) 16
24) 8
25) 4
26) 2
27) 256
28) 255
29) 72
30) 682
31) 24/47
32) 539
33) 485
34) 483 ½
35) 523 2/3
36) 6/13
37) 3/47
38) 24/47
39) 6/47
40) 40/47
41) 16
42) 15/100= 3/20
43) Y
44) 400
45) Y
46) 3280
47) 918
48) 4864
49) 2880000
50) 54,000
51) 2499
52) $32.89
53) 22,496
54) 132,000
55) 1170
56) 1616
57) 76 ¼
58) 146 2/5

Unit 31

1) negative **2)** F (Zero has no reciprocal) **3)** F **4)** T **5a)** 9 9/10 or 9.9 **5b)** 999 999/1000 or 999.999 **c)** 4 4/5 or 4.8 **d)** .04 **e)** 2 2/3 or $2.\overline{6}$

6) 1/1000 or .001 **7)**


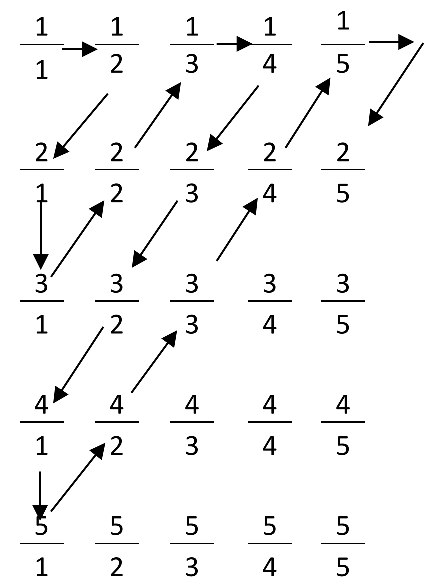
8) 1 1/3 to 2

9) All numbers greater than 1 **10)** 1

QUIZ ANSWERS

Unit 32

1) T **2a)** 1/1000 **b)** $\dfrac{1}{17^{640,000}+1}$ ***c)** greater than, $\dfrac{1}{17^{640,000}}$



4a) T b**)** 399/400 **c)** Yes **5)** Yes, No **6)** Yes (because the matching plan still works.) **7)** $\aleph_0$ or Aleph Null.

Unit 33

1a) $.\overline{1}$ **b)** $.\overline{7}$ **c)** $.\overline{07}$ **d)** $.0\overline{07}$ **e)** $.00\overline{08}$ **f)** $.\overline{70}$ ***g)** $.\overline{13}$ ***h)** $.0\overline{13}$ **i)** $.\overline{142857}$ **j)** $.\overline{428571}$ **k)** $.\overline{27}$ **l)** $.0\overline{857142}$ **m)** $.\overline{09}$ **n)** $.\overline{63}$ **o)** $.0\overline{7}$ **2a)** 4/9 **b)** 77/333 **c)** 3 1/33

d) Note: In $.0\overline{03}$, the repeating part occupies 2 places, so we have 3/99, but it is over one more place, so we have 3/99 $\div$ 10 = 3/990 = 1/330. Check it with your calculator. It should give 0.003030303. . . (= $0.0\overline{03}$). **e)** T **f)** 2/7 **g)** 1/9 $\div$ 100 = 1/900 **h)** 3/700 **i)** 3 6/700 = 3 3/350.

<u>Answers</u> are in **bold print** **<u>and underlined</u>**

3)

Integers $\begin{cases}\text{Negative whole numbers} \\ \underline{\textbf{Zero}} \\ \underline{\textbf{Positive whole numbers}}\end{cases}$

<u>Rational</u> Numbers $\begin{cases}\underline{\textbf{Integers}} \\ \text{Ratios of one integer to another}\end{cases}$

<u>Algebraic</u> Numbers $\begin{cases}\underline{\textbf{Rational Numbers}} \\ \text{Solutions to } \underline{\textbf{Algebraic}}\ \underline{\textbf{Equations}}\end{cases}$

4a) T **b)** T **c)** F

QUIZ ANSWERS

Unit 34

1) 13 **2)** 21/110 **3)** 801/160400 **4)** 19/25 **5)** 60/899 **6)** 35/72 **7)** 119/72 **8)** 145/168
9) 1/224 **10)** 2/297 **11)** 1/8 **12)** 3, 49 **13)** 5, 5 **14)** 1, 7 **15)** 5/6 **16)** T **17a)** 21/110
b) 12/13 **c)** 1/10, 1/11 **18)** 99, 100, 199/9900

Unit 35: No Quiz

Unit 36

1) 45685 **2)** a,b,c,d should all be circled. **3)** $2^{\aleph_0}$ **4)** T **5)** π **,** e**,** probably.

6) Transcendental **7) a)** very slim **8)** No **9)** F 10) T 11) T **12)** Here is a countable set of numbers (exact wording is not important but the sentence should contain a claim of having a countable set or list of numbers). **14)** Cantor ran a diagonal down a presumed complete list of decimals, added 1 to each digit (or made some other consistent change in each), that created a number not already in the list. This provided a false consequence to the hypothesis you stated (?!) in exercise 12. **15)** T **16a)** $2^{\aleph_2} = \aleph_3$

16b) Yes **16c)** No **17) a)** Yes **b)** No **c)** Nothing significant
18) a) disproved **b)** proved

Unit 37: No Quiz

Unit 38

First assume that Set X, the set of sets which are not members of themselves, is itself an X. Note that doing this makes Set X a member of itself, which it is not. So, if X is an X, then it is an S. Second, assume that Set X is an S. This makes X a member of itself, which it is not.

So, If X is an X, then it is an S; but if X is an S, then it is an X.

QUIZ ANSWERS

Unit 39

1a) $\boxed{A}$ $\frac{1}{1}+\frac{1}{2}+\left\{\frac{1}{3}+\frac{1}{4}\right\}+\left\{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right\}+\left\{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right\}+\cdots$

b) $\boxed{B}$ $\underbrace{\frac{1}{1}}+\frac{1}{2}+\underbrace{\left\{\frac{1}{4}+\frac{1}{4}\right\}}+\underbrace{\left\{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right\}}+\underbrace{\left\{\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\right\}}+\cdots$

c). $\boxed{C}$ $\frac{1}{2}\quad\frac{1}{2}+\frac{1}{2}\quad\frac{1}{2}\quad+\quad\frac{1}{2}\quad+\quad\frac{1}{2}$

2a) 1, 1.25, divergent **b)** 2, 2 ½, divergent c**)** convergent d) 1/9

3) T 4) T 5) greater

6) Since C grows slower (or is *less*) than A, and C is divergent, then A must be divergent.

Unit 40

1)

34 x 28	
17 x 56	
8 x 112	
4 x 224	
2 x 448	56
1 x 896	896
	952

2)

17 x 13	
8 x 26	
4 x 52	
2 x 104	13
1 x 208	208
	221

3)

400 x 48	
200 x 96	
100 x 192	
50 x 384	768
25 x 768	6144
12 x 1536	12288
6 x 3072	19200
3 x 6144	
1 x 12288	

4)

32 x 102
16 x 204
8 x 408
4 x 816
2 x 1632
1 x 3264

5) 224 **6)** 221 **7)** 896 **8)** 952 **9)** 3000 **10)** 3200 **11)** 3060 **12)** 3264 **13)** 4200 **14)** 4340

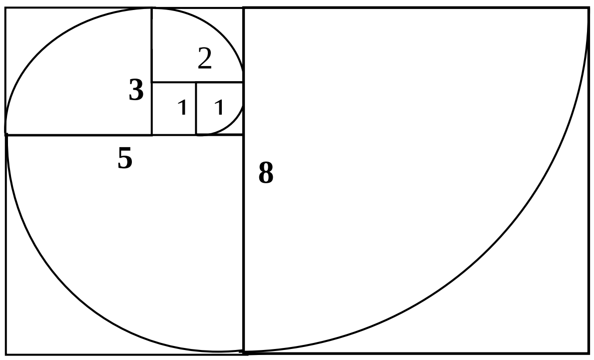
15) 4464 **16)** 4410 **17)** 16 **18)** 25, 24 **19)** 0

QUIZ ANSWERS

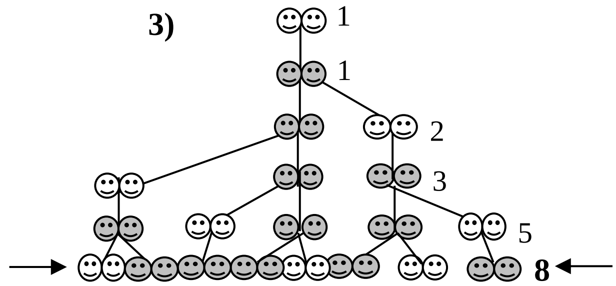
Unit 41

1) 1 **1 2 3** 5 **8 13 21 34**

2)



3)



Your grouping arrangements could be different. Left pair could be exchanged with its parents' other pair, for example.

4)Answers in bold print.

1 **4 6 4** 1

1 5 10 10 5 1

1 6 **15 20 15** 6 1

1 7 **21 35 35 21** 7 1

5a) 1/2 **5b)** 2/3 **5c)** 3/5 **5d)** 5/8 **5e)** 8/13 = $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\frac{1}{1}}}}}}$ **5f)** 13/21 **5g)** 21/34 34/55

Unit 42

1a) 3202 **1b)** 120,002 **1c)** 80,002 **2a)** 106 **2b)** 90,016 **2c)** 180,016 **3a)** 102 **3b)** 63

4) 1,200,000 **5)** 1,020 **6a)** 20 **6b)** 74, that is (18 x 5 – 4 x 4) **6c)** (300 x 18 – 299 x 4 =) 4,204

Unit 43

1) 2 **2)** 19 **3)** 2 **4)** 180 **5)** 0 **6)** 7 **5)** 5 **8)** 6 **9)** -3 **10)** 8 **11)** -8 **12)** -9 **13)** -1 **14)** 3 **15)** -4 **16)** 0 **17)** -1 **18)** 1 **19)** 2 **20)** 1 **21)** 2 **22)** 3 **23)** 3 **24)** 10 **25)** 9 **26)** 0 **27)** 0 **28)** 1 **29)** ½ **30)** 0 **31)** -1 **32)** 1 **33)** 0 **34)** 0

QUIZ ANSWERS

Unit 44

1) 2 **2)** 61/20 **3)**1 **4)** 9 **5)** 1 **6)** 0 **7)** 0 **8)** 121 **9)** 1 **10)** 2 **11)** 2 **12)** 2 **13)** 2 **14)** 0 **15)** 0
16) 1 **17)** 0 **18)** 1 **19)** -1 **20)** -1 **21)** 12 **22)** 1 **23)** 0 **24)** 6 **25)** 2 **26)** 1 **27)** 1 **28)** $1\frac{1}{5}$
29) 1 **30)** 0

Unit 45

1) 6, 4 or 4,6 **2)** 1, 13, or 13, 1 **3)** 6, 6 **4)** 5, 9 **5)** 8, 3 **6)** 54, 2 **7)** 11, 11 **8)** 24, 3 **9)** 11, 10
10) 16, 14 **11)** 8, 6 **12)** 4, 1 **13)** 5, 13 **14)** 1, -1 or -1, 1 **15)** 6, -3 **16)** 13, 7 ***17)** $\sqrt{5}$, $\sqrt{5}$
18) 16, 3 **19)** 8, 4 **20)** 2, ½, or ½, 2

Unit 46

1) $[\,\square\,] = 6$ $\left\{6, 7, 6.5, 5.999\ldots, 3.9999, 2\frac{7}{2}\right\}$

2) $[\,\square\,] = 7$ $\left\{7\frac{6}{7}, 7\frac{8}{7}, 750\%, 7.\overline{9}, 6.5, 6.\overline{9}\right\}$

3) $[\,\square + 8\,] = 13$ $\left\{4, 9, 9.\overline{9}, 4.5, 5.\overline{9}, 5.0001, 5.9999\right\}$

4) $[\,\square\,] = -4$ $\left\{-3.9, -4, -4.1, 4.1, -3.\overline{9}, -3.1, -3\right\}$

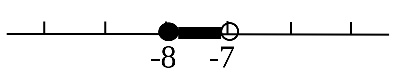
5) $[\,\square\,] \times 7 = 14$ $\left\{2, 3, 1, 2\frac{1}{7}, 2\frac{1}{4}, 2\frac{1}{6}\right\}$

6) $6 + [\,\square\,] = -2$ $\left\{-2, 2, 2\frac{1}{2}, 6, -6, -8, -8\frac{1}{2}, -7\frac{1}{2}, -7, -2\frac{1}{2}, -7.25\right\}$

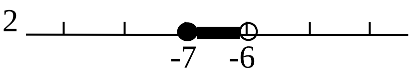
7) $[\,\square \div 5\,] = 4$ $\left\{21.5, 24.5, 25.5, 27.5, 10, 15\frac{1}{2}, 20, 20.3\right\}$

8) $\left[\,\square + \frac{1}{4}\,\right] = \square + \frac{1}{4}$ $\left\{\frac{3}{4}, 1, 1\frac{1}{4}, -1, -\frac{3}{4}, -1\frac{3}{4}, -\frac{1}{4}, 1\frac{3}{4}, -5\frac{1}{4}, 5\frac{3}{4}\right\}$

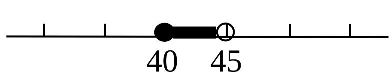
9) $[\,\square\,] = -8$



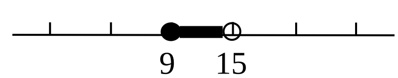
10) $5 + [\,\square\,] = -2$



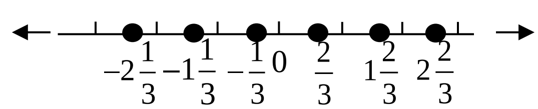
11) $\left[\frac{\square}{5}\right] = 8$



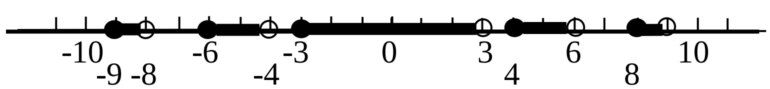
12) $\left[\frac{\square + 3}{6}\right] = 2$

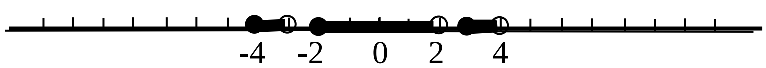


13) Show at least two negatives and two positives on this the graph: $\square + \frac{1}{3} = \left[\,\square + \frac{1}{3}\,\right]$



QUIZ ANSWERS

14) $\left[\frac{\square}{3}\right] = \left[\frac{\square}{4}\right]$ 

15) $\left[\frac{\square}{2}\right] = \left[\frac{\square}{3}\right]$ 

Unit 47

1) 1, 3, 6, 10 15, 21, 28, 36

2) $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\frac{1}{28}+\frac{1}{36}\ldots$

3) $\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}\ldots$

4) $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\ldots$

5) $2 \times \left(\frac{1}{1}\overbrace{-\frac{1}{2}+\frac{1}{2}}\overbrace{-\frac{1}{3}+\frac{1}{3}}\overbrace{-\frac{1}{4}+\frac{1}{4}}\ldots\right)$

6) The Ramanujan series is convergent and its sum is 2.

7) 2, 6, 12, 20, 30, 42, 56, 72. Each number is the product of consecutive numbers.

Unit 48

1) Euler **2a.** NT **b.** NT **c.** T **d.** T **e.** T **f.** NT **g.** T **h.** T **i.** NT **j.** NT **3a.** T **b.** T **c.** T **d.** F **e.** F **f.** T **g.** F, figure may have 0 even vertices; **h.** F, figure may have 0 odd vertices; **i.** T **j.** T **4)** A traceable figure must have exactly 0 or 2 odd vertices.

Unit 49

1) a,c,e,h,i
b,d,g,f,j

2) not a; a **3)** ~a $\wedge$~b **4)** ~(a $\vee$b) **Exercise 5)**

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
a	**b**	**~a**	**~b**	**a $\wedge$ b**	**a $\vee$ b**	**~(a $\wedge$b)**	**~(a $\vee$ b)**	**~a $\vee$~b**	**~a $\wedge$ ~b**
0	**0**	**1**	**1**	**0**	**0**	**1**	**1**	**1**	**1**
0	**1**	**1**	**0**	**0**	**1**	**1**	**0**	**1**	**0**
1	**0**	**0**	**1**	**0**	**1**	**1**	**0**	**1**	**0**
1	**1**	**0**	**0**	**1**	**1**	**0**	**0**	**0**	**0**

6a) their "nots" **b)** **~(r$\wedge$t)** $\longleftrightarrow$ **~r$\vee$ ~t** **c)** True **d)** DeMorgan's law says: The not of the disjunction of two statements is equivalent to the conjunction of the "nots" of the statements.
7a) No **b)** Yes

QUIZ ANSWERS

Unit 50

1)

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20

2)

1	2	3	4	5	6	7	8	9	10
1	3	6	10	15	21	28	36	45	55

3)

```
1 Let Goal = 10,000
2 Let Num = 100
3 Print Num
4 Let Num = Num + 5
5 If Num > Goal then stop
```

Unit 51

1) 123 2**13** **312**

2)

abcd	bacd	cabd	dabc
abdc	badc	cadb	dacb
acbd	bcad	cbad	dbac
acdb	bcda	cbda	dbca
adbc	bdac	cdab	dcab
adcb	bdca	cdba	dcba

3) 3 x 4 x 3 x 5 = 180

4) 362880 **5)** 3628800

6) 3628800 **7)** 5040

8) a) 3628800 **b)** 39916800

9) 13 **10a)** 132 **10b)** 1000 **10c)** 1/10 **11a)** 45 **11b)** 1/45 **11c)** F **11d)** 1/2

Units 52 and 53

1) 24 **2a)** 60 **2b)** 20 **3a)** 720 (= 6!) **3b)** 720 **3c)** 120 **4a)** 50! **4b)** 2450 **4c)** 2500 **5a)** 24

***5b)**

efg	feg	gef	hef
egf	fge	gfe	hfe
efh	feh	geh	heg
ehf	fhe	ghe	hge
egh	fgh	gfh	hfg
ehg	fhg	ghf	hgf

***6)** 21233664 (= 8 x 24 x 24 x 24 x 24 x 8)

***7)** 720 (= 6!)

Unit 54

1a) 2 **b)** 8 **c)** 9 ***d)** 1 **e)** b, d **2a)** Yes, $2\frac{1}{3}$, yes **b)** Yes, 0x8=0 **c)** Yes, $0 \times 2.89\frac{1}{17} = 0$

d) T **3a)** 0,0, yes **b)** 57 ½, 0 ,0, Yes, $0 \div 0$ = any number, so is not acceptable

4) $a \neq b, c = 0$; $a = b$, $c = 0$; $a = b$, $c \neq 0$

5) a ≠ 37 ½% , b = 0; a = 37 ½%, b = 0; a = 37 ½% , b ≠ 0

6) $a \neq 0$, $b = 0$, $a = 0$, $b = 0$; $a = 0$, $b \neq 0$

***7)** $a = 0$, $b \neq 0$; $a = 0$, $b = 0$, not possible to make zero

QUIZ ANSWERS

Units 55 and 56

1a) ab bc ac
ba cb ca
1b) ab bc ac **2a)** 8! or 40,320 **b)** 336 **c)** 56 **3)** $\frac{10\times9\times8}{3\times2\times1}\times\frac{4\times3}{2}=720$

4) 20 **5)** 142,506 **6)** 15 **7)** $\frac{8\times7\times6\times5\times4\times3\times2\times1}{7\times6\times5\times4\times3\times2\times1}=8$, $\frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 7$, $8 \div 7 = 8/7$

8) 1000 **9)** 101 **10)** 1

Unit 57

1)

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

5)

x	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

2) 5 **3)** 2, 2 **4)** 2 **6)** 0
7) 0 **8)** 0 **9)** 0 **10)** 0
11) 0 **12)** 5 **13)** 2 **14)** 1
15) 1 **16)** 5 **17)** 4 **18)** 5
19) 8 **20)** 2 **21)** 3

Unit 58

1a) 5 **b)** 5 **c)** g <u>r</u> <u>u</u> <u>e</u> <u>n</u> <u>t</u>, 5 d) 5 **2a)** 3 **b)** 17 **c)** 5 **d)** 14 **e)** Any two of: 0, 3, 5, 15

3a) 17, 30, 43, 56 . . . **b)** 14, 16, 18, 20 . . . **c)** 114, 127, 140, 153 . . .

4a) N **b)** T **c)** T

Unit 59

1)

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

○	I	V	H	R
I	I	V	H	R
V	V	H	R	I
H	H	R	I	V
R	R	I	V	H

QUIZ ANSWERS

2a) 0, I **b)** 3, V **c)** Think of a diagonal from upper left corner. The patterns of the upper right half is the mirror image of the lower left, giving results like 3 + 1 = 0 and 1 + 3 = 0; that is, 2 + 1 **=** 1 + 2. Also, **V**○ H **=** H○ **V.** So, the operation is commutative for Mod 4 and Flip Cards. Both groups are abelian. (Your answer does not have to be quite so thorough.)

3) H○(V ○ R) = (H○V) ○ R. Any symbols from that set are okay.

4) Closure, Identity element, Inverse, Associative, (Commutative, often)

5) Distributive Property; 2 x (3 + 5) = (2 x 3) + (2 x 5) = 6 +10 = 16

Unit 60

1) 3, 9 **2)** T **3)** 11, 71 **4)** no **5)** no **6a)** 30 or 29 **6b)** yes **7a)7 b)** yes **8)** T

9a) 2^2 x 3^2 x 5 x 7 **b)** 2 x 3^2 x 5 x 11 **10a)** F **b)** T **c)** T **11a)** T **b)** F **c)** T

12a) False **b)** False **c)** False **13a)** 12 **b)** 24 **c)** 60 **d)** 20000 **e)** 28,160 **f)** 576

Unit 61

1a) T **b)** T **c)** F **d)** T **e)** T **f)** F **g)** T **h)** F **i)** F **j)** F **k)** T **l)** F **m)** T **n)** T **o)** F

2a) F **b)** T **c)** F **3a)** T **b)** T **c)** T **d)** T **4a)** 1001! + 3, 1001! + 4, 1001! + 5

4b) 20,000,001! +1, 20,000,001! + 2, ... 20,000,001! + 100, 20,000,001! + 101

5a) 2, 3, 5 **b)** 2, 5 **6a)** No. Consecutive natural numbers do not share factors **6b)** No, because P!+1 is consecutive to P! which already owns all factors less than P.

6c) P, T.

TODAY'S TWISTER

A Daily MathEnriched Program

180 Problems
For Middle Schools, Junior High Schools
And Beyond

by

Paul Dickie

For the Teacher

Years of pleasant and successful experiences with these twisters in the public schools of Concord, Massachusetts, have allowed the evolution of ideas which I think you will find helpful. Three administrative ingredients are recommended:

1. Provide a ready supply of individual twister copies in a convenient place for student pickup when entering the classroom. A pocket fashioned from light cardboard like file folder material and stapled to tack board is one way to make twisters easily available.
2. Keep participation optional with no bearing on math grade.
3. Maintain a public record of achievement on a posted chart showing the current number of twisters solved by each student. Students are very willing to be designated to do this.

The supply of twisters, a simple deposit box with a slit on top and the public record ideally would be in one area. Deadline for return should be understood; the beginning of next class is usually convenient. Post at least one student's twister from the previous time, more when clever alternative solutions have been given. Students like to see their own on display. Latent math talent can surface in this program of twisters.

The program runs itself to a large extent. Correcting and public record keeping should not be time consuming, especially when done by a designated as noted above. Your clerical tasks should be limited to using the copy machine and paper cutter. You might even find or train a student for copy work. You will want to correct some twisters yourself because of the need to judge brief narrative answers and alternative solutions.

It should not be necessary to "sell" the program other than to introduce and describe it and provide a reasonably attractive display. Emphasize that participation is optional.

Twisters could be given two or three times a week rather than the usual daily, allowing a wider choice from the collection. They vary considerably in difficulty. Very few are truly easy.

The order is not significant except in a few obvious cases where definitions such as "factorial" and "greatest integer" function are given and used again later.

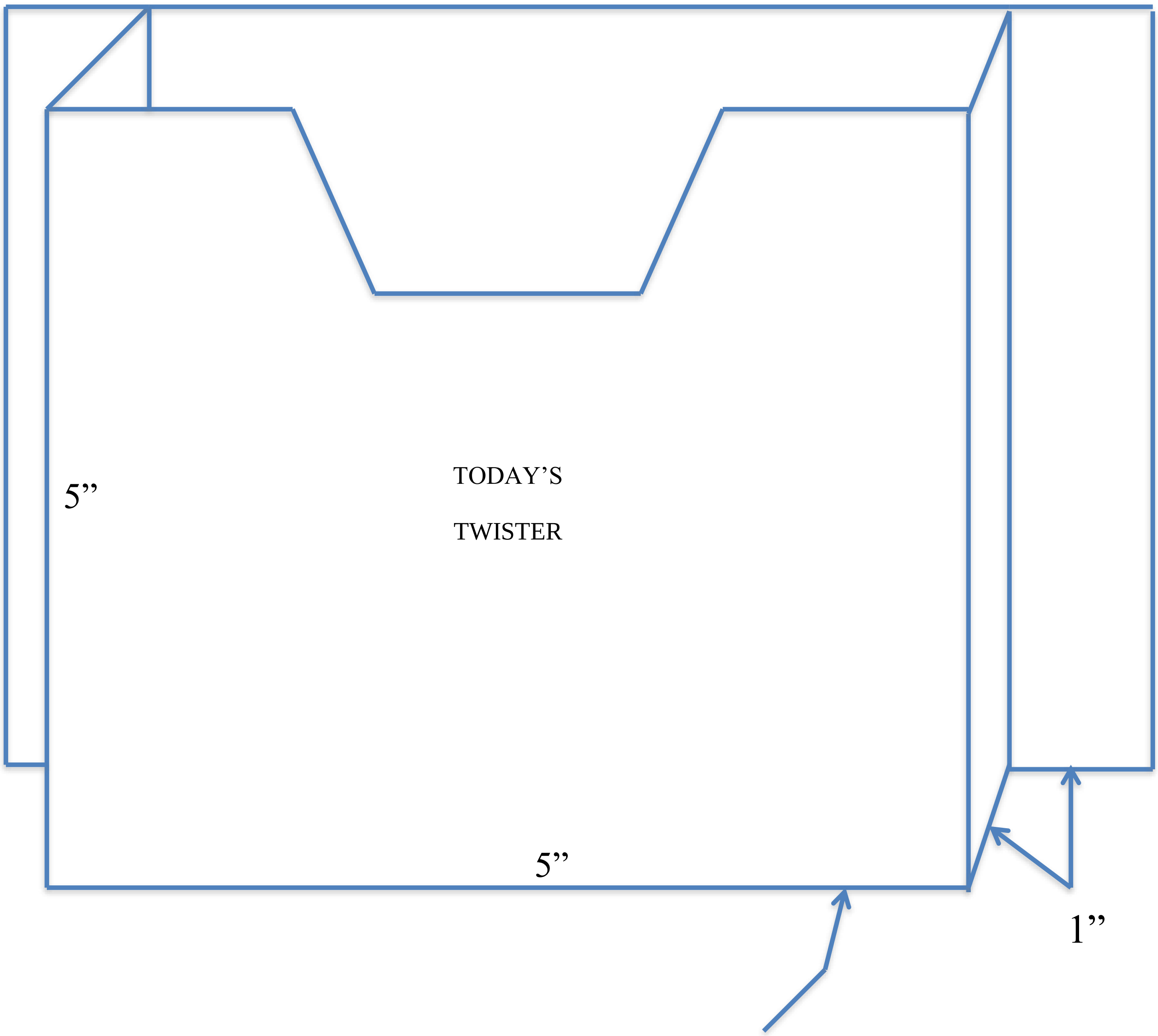
Knowledge of algebra is not assumed. The two or three twisters using number bases may be skipped if students have not studied this topic. Almost everything else is accessible to students beyond grade five but some are quite difficult. Negative numbers are used two or three times. You will recognize some "old chestnuts" but most are original and some have been submitted by students, a practice you might encourage.

This collection represents much creating, searching and sifting. Good luck. I hope these twisters will do for your math classes what they did for mine: add a gratifying aspect of interest and enthusiasm.

Paul G. Dickie

Wolfeboro, NH

Here is a diagram of a pocket you can make to display the daily twisters. Every visible surface except the inner back belongs to the front of the file folder.



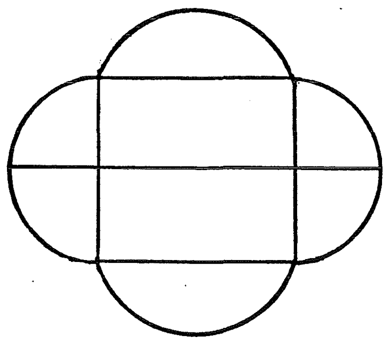
Underneath is the "spine" of a file folder, 1". It is folded, cut and stapled to a tack board to display up to a 1" thick stack of twisters.

1. TODAY'S TWISTER

NAME ______________________________

Put the numbers 1 - 8 into the boxes so that each number is used once and there are no adjoining boxes with consecutive numbers.

(They may not even join at corners.)



2. TODAY'S TWISTER

NAME ______________________________

What number has its half, its double, and its third add up to 68?

ANSWER: ______________________________

3. TODAY'S TWISTER

NAME ______________________________

Name the two numbers which have a sum of 221 and a difference of 39.

ANSWER: ______________________________

4. TODAY'S TWISTER

NAME ______________________________

Sample: Let: [1.8] = mean "Give the largest integer" in 1.8. Answer: 1.

Now give the answer to: (don't work it out!!!)

[(834/835 + 1/836)] =

ANSWER: ______________________________

5. TODAY'S TWISTER

NAME ________________________________

There is a queen just below a queen.

There is a jack just above a queen.

There is a spade just above a spade.

There is a heart just below a spade.

1
2
3

Name the cards:

Card 1 ____________________

Card 2 ____________________

Card 3 ____________________

6. TODAY'S TWISTER

NAME ________________________________

A boy had two urns; one held exactly three quarts and the other exactly five quarts. His father sent him to the well with instructions to return with exactly four quarts of water.

The sides of the urns were not marked in any way. How did the boy manage to measure exactly four quarts of water?

ANSWER: ________________________________

7. TODAY'S TWISTER

NAME ________________________________

Ann found two dollars. She then had five times as much money as she would have had if she lost two dollars.

How much money did Ann have originally?

ANSWER: ________________________________

8. TODAY'S TWISTER

NAME ________________________________

Sample: Write three sixes to make six.

Answer: 6 x 6/6

Write three sixes to make seven.

ANSWER: ________________________________

Write four fours to make five. You may use basic operations, a square root symbol, parentheses, decimal points and exponents (as long as the exponent is a 4).

ANSWER: ________________________________

9. TODAY'S TWISTER

NAME ___________________________________

The number 9 is an intriguing one as you will discover as you fill in the blanks.

(0 X 9)+ 1 = _

(1 X 9) + 2 = __

(12 X 9) + 3 = ___

(123 X 9) + 4 = ____

(1234 X 9) + 5 = _____

(12345 X 9) + 6 = ______

(123456 X 9) + 7 = _______

(1234567 X 9) + 8 = ________

(12345678 X 9) + 9 = _________

(123456789 X 9) +10 = __________

TOTAL= ____________

10. TODAY'S TWISTER

NAME ___________________________________

Simplify:

$$10 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 - \frac{1}{3}}}}}$$

Hint: Start at the bottom and work up.

ANSWER: ______________________________

11. TODAY'S TWISTER

NAME ___________________________________

A perfect number is a number whose factors add up to that same number. The factors should not include the number itself in this case. The factors of a number include all those numbers which divide into it evenly.

The factors of 6 (excluding 6 itself) are 1, 2 and 3. Also, 1 + 2 + 3 = 6, so 6 is a perfect number.

There is another perfect number between 15 and 30. Find it.

ANSWER: ______________________________

12. TODAY'S TWISTER

NAME ___________________________________

Into how many quarter inch cubes may a one inch cube be cut?

ANSWER: ______________________________

13. TODAY'S TWISTER

NAME ______________________________

What number may be placed in both boxes to make this true? (Same number in both boxes)

$$\frac{2}{\square} = \frac{\square}{450}$$

ANSWER: ______________________________

14. TODAY'S TWISTER

NAME ______________________________

In a list of four numbers the average of the first two is 10, the third number is 8, and the average of the four numbers is 8.

What is the fourth number?

ANSWER: ______________________________

15. TODAY'S TWISTER

NAME ______________________________

The reciprocal of any number, say n, is 1/n (except when n = 0).

By exactly how much does 1.10 exceed its reciprocal?

ANSWER: ______________________________

16. TODAY'S TWISTER

NAME ______________________________

Move one number from one group to another so that the sum of the numbers in each group will then be equal.

(Answer by using arrows or a description of what to do on back.)

1, 2, 3	4, 5, 6	7, 8, 9

ANSWER: ______________________________

17. TODAY'S TWISTER

NAME ________________________________

LIST AND ADD all divisors of 220 including 1 but not including 220.

(Note: For your final answer give only the sum of the divisors.)

FINAL ANSWER: ____________________________

18. TODAY'S TWISTER

NAME ________________________________

Harry and Cary were counting their money. Harry said, "Give me one of your dollars and I'll have as many as you."

"Yes", Cary said.

"But if you give me one of your dollars then I'll have twice as many as you have."

Originally, how many dollars did each person have?

ANSWER Harry: ____________________________

ANSWER Cary: ____________________________

19. TODAY'S TWISTER

NAME ________________________________

Mr. Barcatchem, carrying his rifle, walks directly south from a point, traveling a distance of three miles. He then walks east for three miles and at that point shoots a bear.

Upon checking his position, he finds that he is still three miles from where he started.

What color is the bear?

ANSWER: ________________________________

20. TODAY'S TWISTER

NAME ________________________________

If an egg and bacon cost 90¢ and the bacon cost 70¢ more than the egg, how much does the bacon cost?

ANSWER: ________________________________

21. TODAY'S TWISTER

NAME ____________________________________

Someone wants to build a square-shaped house with windows on every side, each window having a view to the south.

How may this be done?

ANSWER: ________________________________

22. TODAY'S TWISTER

NAME ____________________________________

Find two numbers whose difference and whose quotient are both equal to three.

ANSWER: ________________________________

23. TODAY'S TWISTER

NAME ____________________________________

Simply by looking and thinking, tell which of the following could not possibly be the square of an integer:

123454321 345676543
3086358025 4444355556
1111088889

ANSWER: ________________________________

24. TODAY'S TWISTER

NAME ____________________________________

Suppose that a pond lily doubles itself every day and at the end of 37 days has half-filled a 10,000 acre lake.

How many days from the start does it take to fill the lake?

ANSWER: ________________________________

25. TODAY'S TWISTER

NAME ________________________________

A prime number is divisible by no other number except one. Examples are 5, 7, 19, 41, etc…

Name:

a. The only even prime ____________________

b. A prime between 25 and 30 ______________

c. A prime between 50 and 55 ______________

d. All the primes in the nineties ______________

ANSWER: ______________________________

26. TODAY'S TWISTER

NAME ________________________________

One mathematician said "Every even number can be expressed as the sum of two primes".

Prove that this is true for each of these even numbers. That is, write the sum of two primes for each even number below. The first one is done as an example.

18 13 + 5 (or 11 + 7)

14 ______________________

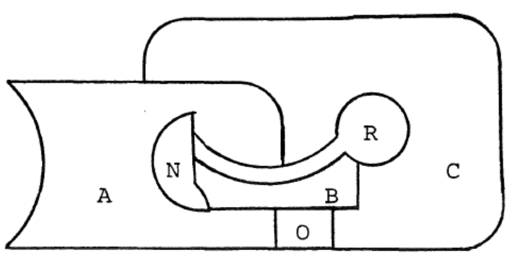
20 ______________________

100 ______________________

27. TODAY'S TWISTER

NAME ________________________________

Below is a "map" of six countries. Only four colors may be used to color the six countries and no bordering countries may have the same color. Which of the pairs of countries listed could possibly have the same color? (Careful - only one answer is possible.)

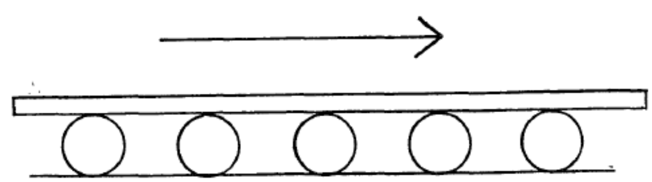


A & O   A & C   N & O   N & R   O & R

ANSWER: ______________________________

28. TODAY'S TWISTER

NAME ________________________________



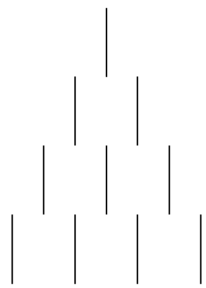
A plank is pushed forward on rollers as shown. How far does the plank advance if the rollers advance two feet?

ANSWER: ______________________________

29. TODAY'S TWISTER

NAME ____

Below is a set of pins at the end of a bowling alley designed to be struck from the "top". Cross out and rearrange the position of three pins so the group is designed to be struck from the "bottom".



30. TODAY'S TWISTER

NAME ____

Find the next number in the series.

4, 9, 17, 35, 69, ____

ANSWER: ______________________________

31. TODAY'S TWISTER

NAME ____

Try to guess the rule and use it to finish the last problems.

3, 1	11
7, 3	55
12, 10	164
10, 12	____
9, 2 ½	____
1, 5	____
3, 0	____
4, __	22
__, 20	45

ANSWER: ______________________________

32. TODAY'S TWISTER

NAME ____

A cryptarithm is an ordinary arithmetic problem disguised by using letters instead of numbers. You must discover what numbers are being replaced by letters.

What numbers for L, M, and N will make this addition example work?

$$\begin{array}{cccc} & L & M & N \\ + & 6 & L & M \\ \hline N & 5 & L & 0 \end{array}$$

ANSWER: L = _____ M = _____ N = _____

33. TODAY'S TWISTER

NAME ____

Braithenwald walks for three days at the rate of 12 miles per day and for 5 days at the rate of 8 miles per day.

What is his average rate of walking in miles per day?

ANSWER: ________________________________

34. TODAY'S TWISTER

NAME ____

Eight persons were in a room. Each person shook hands with each other person. How many handshakes occurred?

ANSWER: ________________________________

35. TODAY'S TWISTER

NAME ____

The average (mean) of 40 and 60 is 50. What is the average of 1/40 and 1/60?

ANSWER: ________________________________

36. TODAY'S TWISTER

NAME ____

A piece of paper is colored as follows: 1/3 of it is red, 1/4 of it is blue, and the remaining 8 square inches are burgundy.

What was the area of the original piece of paper?

ANSWER: ________________________________

37. TODAY'S TWISTER

NAME ____

Harry bought a table tennis paddle for $1.30 and sold it to Zinger for $1.30. Zinger sold it back to Harry for $1.20. Next, Harry sold the paddle again for $1.35.

How much did he make on the whole transaction?

ANSWER: ______________________________

38. TODAY'S TWISTER

NAME ____

How many times does the clock strike during a day if it strikes the correct number of times for each hour?

ANSWER: ______________________________

39. TODAY'S TWISTER

NAME ____

A man planted ten trees. He had three rows of four trees each. How did he do it?
Draw your answer.

ANSWER: ______________________________

40. TODAY'S TWISTER

NAME ____

In a line of girls there were 2 girls in front of a girl, 2 girls behind a girl, and there was a girl in the middle.
How many girls were there (minimum number)?

ANSWER: ______________________________

41. TODAY'S TWISTER

NAME ______________________________

There are four tents at a carnival and Betsy enters each one. It costs $1 to enter a tent and $1 to exit. Inside each tent Betsy spends half of what is in her pocket. When she emerges from the last tent she has no money left.

How much did she have at the beginning?

ANSWER: ______________________________

42. TODAY'S TWISTER

NAME ______________________________

If 23 (base ?) = 15 (base ten), then what numeration system is the 23 written in?

ANSWER: ______________________________

43. TODAY'S TWISTER

NAME ______________________________

Janis determined that it took six seconds for the town hall clock to strike 6. How long will it take this clock to strike 12?

ANSWER: ______________________________

44. TODAY'S TWISTER

NAME ______________________________

It is now 11:30. When will the hands of the clock next be at right angles with each other?

a. A little before 11:45

b. Exactly quarter of twelve

c. A little after quarter of twelve

d. Exactly 12:15

ANSWER: ______________________________

45. TODAY'S TWISTER

NAME ____

If you take this number and triple it, then add 6, and finally double the result, the answer is 57.
What is the number?

ANSWER: ______________________________

46. TODAY'S TWISTER

NAME ____

Write the next three numbers for each sequence.

a. 3, 8, 13, 18, ___, ___, ___

b. 0, 1, 3, 6, 10, ___, ___, ___

c. 0, 1, 3, 7, 15, 31, ___, ___, ___

d. 1, 5, 14, 30, ___, ___, ___

47. TODAY'S TWISTER

NAME ____

GROUND RULE: Same shape must contain the same number.

$$2 \times \square + \hexagon = 13$$

$$\hexagon - \square = 4$$

ANSWER:

$\square$ = ______

$\hexagon$ = ______

48. TODAY'S TWISTER

NAME ____

A woman is 46 years old when her daughter is 18. How old will the woman be when she is twice her daughter's age?

ANSWER: ______________________________

49. TODAY'S TWISTER

NAME ________________________________

If each person in a room shakes hands with each other person exactly once, how many people are in the room if there are 15 handshakes?

ANSWER: ________________________________

50. TODAY'S TWISTER

NAME ________________________________

If you wrapped a piece of string around the earth exactly once and then made it one foot longer, which of the following would be true?

a. You could barely slip a piece of cardboard under the string.
b. You could just about fit an orange under the string.
c. You could drive a car under the string.

ANSWER: ________________________________

51. TODAY'S TWISTER

NAME ________________________________

A small purse is full of coins. If you count them by 2's, 3's, or 5's there will be 1 left over. If you count them by 7's there will be none left over. How many coins are in the purse?

ANSWER: ________________________________

52. TODAY'S TWISTER

NAME ________________________________

The perimeter of Julie's rectangular garden is 48 feet. What are its length and width if its area is 140 square feet?

ANSWER: Length ______ft.

Width _______ft

53. TODAY'S TWISTER

NAME ______________________________________

Time can be strange. To what does the following refer?

I occur twice in two seconds, once in a fortnight, but not ever in a century. What am I?

ANSWER: ____________________________

54. TODAY'S TWISTER

NAME ______________________________________

If three boys can wash three cars in three hours, how long will it take four boys to wash eight cars?

ANSWER: ____________________________

55. TODAY'S TWISTER

NAME ______________________________________

Bobby lived on Ronald Road. When asked what the number of his house was he replied, "It has two digits, and when a decimal point is placed between them, the resulting number is the average of the digits in my house number".

What was Bobby's house number?

ANSWER: ____________________________

56. TODAY'S TWISTER

NAME ______________________________________

Discover what Rule is $\xrightarrow{a}$ doing to the left number to get the right-hand answer and fill the blanks.

$5 \xrightarrow{a} 22$

$0 \xrightarrow{a} -3$

$2 \xrightarrow{a} 1$

$3 \xrightarrow{a} 6$

$1 \xrightarrow{a}$ ___

$10 \xrightarrow{a}$ ___

___ $\xrightarrow{a} 61$

___ $\xrightarrow{a} 13$

57. TODAY'S TWISTER

NAME ___________________________________

Two bicycle riders are 25 miles apart. One is traveling 15 mile per hour and the other 10 miles per hour. A bee starts from the front tip of the handlebars of one bike and flies to the front tip of the handlebars of the other and then back and forth from tip to tip, always flying at the rate of 20 miles per hour as the bicycles approach each other.

How far did the bee fly before the cyclists met?

ANSWER: ______________________________

58. TODAY'S TWISTER

NAME ___________________________________

Suppose that the following statement is true: "If Harry is happy then Sally is sad."

What definite conclusion can you draw (if any) if:

a. Harry is happy.
b. Harry is not happy.
c. Sally is sad.
d. Sally is not sad.

ANSWERS:

a. ____________________

b. ____________________

c. ____________________

d. ____________________

59. TODAY'S TWISTER

NAME ___________________________________

A fence 20 feet long requires four posts. How long would a similar fence be which contains 10 posts?

ANSWER: ______________________________

60. TODAY'S TWISTER

NAME ___________________________________

12 3 4 5 6 7 8 9

There are different ways in which the above numbers can be connected to total 99 without changing their order. One solution is:

$$1 + 23 + 45 + 6 + 7 + 8 + 9$$

Try to find another which, like the example given, uses only addition.

ANSWER: ______________________________

61. TODAY'S TWISTER

NAME ____________________________________

Use the number 3 five times to form the number 31.

ANSWER: ______________________________

62. TODAY'S TWISTER

NAME ____________________________________

A circular target with concentric rings has the following numbers printed on it, one in each ring:

16, 17, 23, 24, 36, 40

Tell how exactly 100 could be scored using no more than 6 arrows.

ANSWER: ______________________________

63. TODAY'S TWISTER

NAME ____________________________________

a. If 13.2 (base y) = $9\frac{1}{3}$ (base ten), what number is represented by y?

b. If 0.13 (base n) = $\frac{4}{27}$ (base ten), then n = _____

ANSWER: ______________________________

64. TODAY'S TWISTER

NAME ____________________________________

Give the letter of each which is <u>not</u> divisible by 4:

a. 4^{30} b. 30^4 c. 4^{35} d. 35^4

e. $17^{50} + 17^{50} + 17^{50} + 17^{50}$

f. 17^{50} g. $4^{(20-2)}$ h. $4^{20} - 2$

ANSWER: ______________________________

65. TODAY'S TWISTER

NAME ______________________________

If 4* = 6, 9* = 16, and 50* = 98,

then 2 + (3)* + (2 + 3)* = ??

ANSWER: ______________________________

66. TODAY'S TWISTER

NAME ______________________________

How many buses will I pass coming from the opposite direction when I go from bus stop A to bus stop B if busses leave each station every ten minutes and the trip takes one hour?
Count the bus pulling into A as I leave and leaving B as I arrive.

ANSWER: ______________________________

67. TODAY'S TWISTER

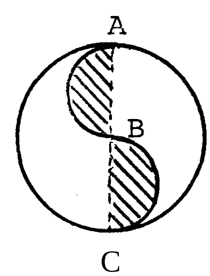
NAME ______________________________

A bus left Atlanta, Georgia, at 4:00 A.M. and drove toward New York City at 55 miles per hour. At 5:00 A.M. the same day a bus left New York City and drove toward Atlanta at 50 miles per hour. At the instant that the drivers passes each other, which one was closer to New York?

ANSWER: ______________________________

68. TODAY'S TWISTER

NAME ______________________________



ABC is a diameter of the large circle with B as its center. Find the area of the shaded S - shaped figure if the area of the large circle is 88 sq. units.

ANSWER: ______________________________

69. TODAY'S TWISTER

NAME ____

If a driver had increased his speed by one-fourth, he would have made his trip in 16 hours. As it was, how long did it take him to make the trip?

ANSWER: ______________________________

70. TODAY'S TWISTER

NAME ____

Write 30 using three equal single digits and any basic operation symbols, including exponents

ANSWER: ______________________________

71. TODAY'S TWISTER

NAME ____

If there are 16 stations on a railroad, how many different tickets are required to connect every station with every other station?

ANSWER: ______________________________

72. TODAY'S TWISTER

NAME ____

If a woman buys a half dozen eggs each week for two weeks and then one dozen each week for three weeks, on the average, how many eggs does she use per week?

ANSWER: ______________________________

73. TODAY'S TWISTER

NAME ______________________________

Three men at 6:00 A.M. started to dig two holes. One of them, worked alone, completed his hole, 3x3x3, in one hour. The other two, each working at the same speed as the first made their hole 6x6x6.
How long did it take them?

ANSWER: ______________________________

74. TODAY'S TWISTER

NAME ______________________________

Remembering that:

Odd no. + even no. = odd no.

Odd no. + odd no. = even no.

Even no. + even no. = even no.

What are the chances, given a pair of identical twelve sided dice with any consecutive 12 numbers printed on their faces, that a total which is an even number will be rolled?

ANSWER: ______________________________

75. TODAY'S TWISTER

NAME ______________________________

Julie's garden with perimeter 54 ft. and width 12 ft. is to have posts put around it 3 feet away from the garden's edge on all sides. They are also to be three feet apart. How many posts will she need?

ANSWER: ______________________________

76. TODAY'S TWISTER

NAME ______________________________

Use four 7's to express the number 87.

Sample: 77/7 - 7 = 4 (not right for 87)

Decimal points and parentheses may also be used.

ANSWER: ______________________________

77. TODAY'S TWISTER

NAME ______________________________

Eleventeen and 2/3 of eleventeen is what part of 7/3 of eleventeen?

ANSWER: ______________________________

78. TODAY'S TWISTER

NAME ______________________________

Some very tricky conclusions about repeating decimals!

True or false

a. $.2 + .7 = .9$ _______
b. $.\overline{2} + .\overline{7} = .\overline{9}$ _______
c. $2/9 + 7/9 = 9/9$ _______
d. $9/9 = 1$ _______
e. $.\overline{9} = 1$ (exactly) _______

ANSWER: ______________________________

79. TODAY'S TWISTER

NAME ______________________________

Janis says that three men can build a garage in 4 days, and if it takes ten boys to do the work of four men, how long will it take two men and three boys to build the garage?

ANSWER: ______________________________

80. TODAY'S TWISTER

NAME ______________________________

Roger noted that there is a cube which has the same number of square units in its surface area as it has cubic units in its volume. How many units in the length of the edge of the cube?

ANSWER: ______________________________

81. TODAY'S TWISTER

NAME ________________________________

Leslie got up early one morning to deliver newspapers. It was still dark outside and none of the lights in the room would work. Leslie knew that there were 10 red socks and 4 white socks in the dresser drawer.

What is the minimum number of socks that Leslie has to pull out of the drawer in the dark to be sure that there are at least two that match?

ANSWER: ________________________________

82. TODAY'S TWISTER

NAME ________________________________

If a goose and a half can lay an egg and a half in a day and a half, how many eggs can seven geese lay in six days?

ANSWER: ________________________________

83. TODAY'S TWISTER

NAME ________________________________

Connect these nine dots with only four straight lines without lifting your pencil from the paper.

Practice on another paper.

0 0 0

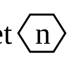
0 0 0

0 0 0

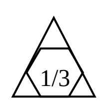
ANSWER: ________________________________

84. TODAY'S TWISTER

NAME ________________________________

Let  mean 2xn – 3 and let  mean $n^2 + 1$.

What value of y will make the following true?



X y = 1

ANSWER: ________________________________

85. TODAY'S TWISTER

NAME ____________________________________

Two cowboys were complaining about their horses, each insisting that his own horse was slower than his partner's. To settle the matter they agreed to a race in which the horse to cross the finish line last would be declared the winner, having proved to be the slower. They started to race across the Arizona desert but soon slowed to a complete stop, each rider being reluctant to prove himself wrong. They dismounted and explained their dilemma to a passing man who made a suggestion, whereupon the men immediately mounted and raced for the finish line. What was the suggestion?

ANSWER: ________________________________

86. TODAY'S TWISTER

NAME ____________________________________

List each number which will make this sentence true (same number in all boxes at one time).

$$\frac{2}{\square - 2} = \frac{\square}{\square - 2}$$

ANSWER: ________________________________

87. TODAY'S TWISTER

NAME ____________________________________

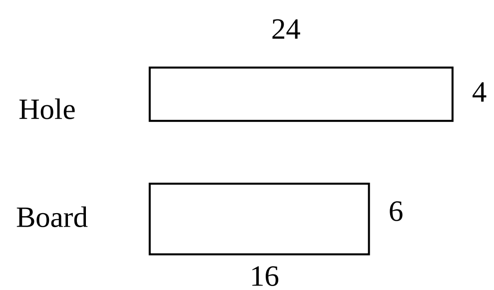
In the bottom of a well 33 feet deep there was a frog who began to travel toward the top. In his journey he ascended three feet each day and slipped back two feet each night. In how many days did he get out of the well?

ANSWER: ________________________________

88. TODAY'S TWISTER

NAME ____________________________________

How can this board be cut into two pieces so it will exactly cover the hole? Draw the cut line and label its dimensions.



ANSWER: ________________________________

89. TODAY'S TWISTER

NAME ________________________________

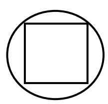
Let n mean $(n + 2)^2$,

Let n mean n + 2, and

Let $\boxed{n}$ mean n – 5

$\boxed{\text{Note: } -3 \times -3 = 9}$

Enclose (-1) within three shapes, not necessarily different shapes, so as to produce the largest possible results and tell what that result is. Example (not the right one!):



-1 = 4

Hint: It's possible to produce a value much larger than 500.

ANSWER: _____________ produces __________

90. TODAY'S TWISTER

NAME ________________________________

What number is just as much less than 92 as its triple is more than 92?

ANSWER: ________________________________

91. TODAY'S TWISTER

NAME ________________________________

4! (called "4 factorial") = 4x3x2x1 = 24

10! = 10 x 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 = 3628800

Compute:

a. 8!

b. $\frac{98!}{96!}$ (Try to think of a short cut.)

ANSWERS: a.______________ b._______________

92. TODAY'S TWISTER

NAME ________________________________

Name each whole number less than 100 that is a perfect square and is also a perfect cube.

ANSWER: ________________________________

93. TODAY'S TWISTER

NAME ____

How many inches in the circumference of a circle whose area is 154 square inches?

Use pi = $^{22}/_{7}$

ANSWER: ______________________________

94. TODAY'S TWISTER

NAME ____

Compute:

a. $(2^3)^{(2^1)}$

b. $(2^{(3^{(2^1)}})$

ANSWER: ______________________________

95. TODAY'S TWISTER

NAME ____

Braithenwald deposited $1 in January, $2 in February, $4 in March, $8 in April, etc.

a. How much money will he have after his December deposit?
b. If he continues, what will be the amount of the 13th deposit in January of the second year?
c. How much money will he then have?

ANSWERS: a.________ b.________ c._________

96. TODAY'S TWISTER

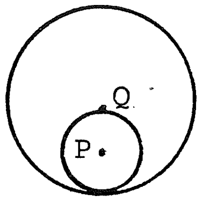
NAME ____

Two freight trains are traveling in opposite directions, one east at 45 miles per hour and the other west at 60 miles per hour.A man on the east-bound train is running west along the tops of the cars at the rate of 10 miles per hour and a man on the westbound train is running west at the rate of 15 miles per hour.At what rate do the two men pass each other when the trains pass?

ANSWER: ______________________________

97. TODAY'S TWISTER

NAME ________________________________



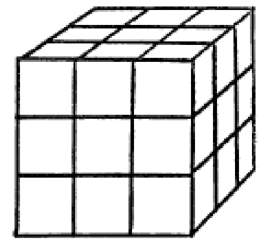
Q is the center of the large circle and p is the center of the small circle. How far does point P travel while the small circle runs once around the inside of the large one if the circumference of the large circle is 22 inches?

ANSWER: ________________________________

98. TODAY'S TWISTER

NAME ________________________________

Here is a three inch cube which has been painted red. It is then cut into one inch cubes along the lines indicated. ·How many of the resulting one inch cubes have exactly:



a. 6 paintedsurfaces?_____
b. 5 painted surfaces?_____
c. 4 painted surfaces?_____
d. 3 painted surfaces?_____
e. 2 painted surfaces?_____
f. 1 painted surface?_____
g. 0 painted surfaces?_____

99. TODAY'S TWISTER

NAME ________________________________

Which one of the following list of inequalities should be left out so the remaining four will contain no contradictions?

$a > b$

$a > d$

$b > c$

$c > a$

$d > c$

ANSWER: ________________________________

100. TODAY'S TWISTER

NAME ________________________________

Evaluate this continued fraction:

$$2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + 2}}}}$$

ANSWER: ________________________________

101. TODAY'S TWISTER

NAME ____

$\sqrt{25} = 5, \quad \sqrt{36} = 6 \quad \sqrt{100} = 10$

4! = 4 x 3 x 2 x 1, 9! = 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1

(Remember factorials?)

Simplify:

$$\left(4\,x\,\sqrt{\frac{36\,x\,99!}{100!}}\,x\,\;\frac{5}{3}\right)!$$

Note: In 36 x 99!, the ! applies only to the 99.

ANSWER: ________________________________

102. TODAY'S TWISTER

NAME ____

A fish is 6 inches long plus two-thirds of its length. How long is it?

ANSWER: ________________________________

103. TODAY'S TWISTER

NAME ____

A field is owned by three people. A has three-fifths of it and B has twice as much as C. What fraction of the field belongs to C?

ANSWER: ________________________________

104. TODAY'S TWISTER

NAME ____

The Big Indian and the Little One. A big Indian and a little Indian were sitting on a fence. The little Indian was the big Indian's son but the big Indian was not the little Indian's father. How could this be?

ANSWER: ________________________________

105. TODAY'S TWISTER

NAME ______________________________

Suppose the pond ponger reproduces by dividing in two every day. On the first day there is one, on the second day 2, the third day 4, etc. If, starting with one pond ponger, it takes 20 days to cover a certa1n area, how long will it take to cover the same area starting with two pongers?

ANSWER: ______________________________

106. TODAY'S TWISTER

NAME ______________________________

As the number in the box gets larger and larger, what value does the complex fraction approach?

$$\frac{9+\frac{2}{\square}}{3-\frac{1}{\square}}$$

ANSWER: ______________________________

107. TODAY'S TWISTER

NAME ______________________________

Let 8_4 mean 8 + 9 + 10 + 11 (4 terms), and 5_3 mean 5 + 6 + 7 (3 terms).

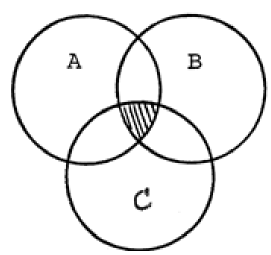
Evaluate:

$$21{,}408_5 - 21{,}406_5$$

ANSWER: ______________________________

108. TODAY'S TWISTER

NAME ______________________________



Set A contains all the two digit whole numbers.
Set B contains all the primes.
Set C contains all whole numbers whose final digit is 1.

List the numbers which belong to the shaded area.

ANSWER: ______________________________

109. TODAY'S TWISTER

NAME ____

If four boys can wash four cars in six hours, how long will it take three boys to wash six cars?

ANSWER: ______________________________

110. TODAY'S TWISTER

NAME ____

If Mary weighs 112 lbs. and Ann weighs 100 lbs., for every 18 lbs. that Mary weighs, there are how many pounds in Ann's weight?

ANSWER: ______________________________

111. TODAY'S TWISTER

NAME ____

Sue wrote the sum of the numbers from 1 to 100.

$$\begin{array}{r} 1 + 2 + \ldots + 98 + 99 + 100 \\ \underline{100 + 99 + \ldots + 3 + 2 + 1} \end{array}$$

From this she was able to find a short cut for adding up all the whole numbers from 1 to 100. See if you can do it.

ANSWER: ______________________________

112. TODAY'S TWISTER

NAME ____

Braithenwald spent 1/3 of his money for candy, ¼ of his money for soda, and 1/5 of his money for ice cream. If he then had 26 cents left, how much did he have at the start?

ANSWER: ______________________________

113. TODAY'S TWISTER

NAME ______________________________________

Boxes of cookies are arranged as shown below in a square pattern. Stacks of seven alternate with single boxes. A clerk in the store sold four boxes and then rearranged the stacking, keeping the square pattern and also keeping nine boxes on each side of the square. Show how he did it.

1	7	1
7		7
1	7	1

ANSWER:______________________________________

114. TODAY'S TWISTER

NAME ______________________________________

Subtract four thousand fourteen hundred and one-half from thirteen thousand thirteen hundred thirteen and one-half.

ANSWER: ______________________________________

115. TODAY'S TWISTER

NAME ______________________________________

If a match and a half costs a penny and a half, how much will 11 matches cost?

ANSWER: ______________________________________

116. TODAY'S TWISTER

NAME ______________________________________

If I add 1000 to a certain whole number, the result is more than if I had multiplied that· number by 1000.

Name all of the numbers for which this is true.

ANSWERS: ______________________________________

117. TODAY'S TWISTER

NAME ________________________________

Name all the positive or negative whole numbers which, when multiplied by themselves, are equal to two more than themselves.

ANSWER: ________________________________

118. TODAY'S TWISTER

NAME ________________________________

Make a Magic Square by putting in the numbers 1 - 9 so that the sum of the horizontals, the verticals and the diagonals equals 15.

ANSWER: ________________________________
____

119. TODAY'S TWISTER

NAME ________________________________

A boy bought a bat and a ball for $1.25. If the bat cost 25 cents more than the ball, how much did each cost?

ANSWER: ________________________________

120. TODAY'S TWISTER

NAME ________________________________

If it takes seven seconds for a clock to strike seven, how many seconds does it take to strike ten?

ANSWER:________________________________

121. TODAY'S TWISTER

NAME ________________________________

This one takes patience. There is a solution which involves only addition (no exponents, square root, subtraction, etc.).

Use eight 8's to make 1000.

ANSWER: ________________________________

122. TODAY'S TWISTER

NAME ________________________________

Let (8, 11) mean the interval of all numbers from 8 to 11 but including neither the 8 nor the 11. Some members are 8 1/3, 9.3067, 10 1/2, 10.89, etc.

How many multiples of 6 are members of the interval:

(200022, 200844)?

Use some ingenuity; don't just count!

ANSWER: ________________________________

123. TODAY'S TWISTER

NAME ________________________________

A farmer was asked whether he had a score of pigs. He said that he did not but if he had as many more, and half that many more, plus two pigs and a half, he would then have a score.

How many pigs did he have?

ANSWER: ________________________________

124. TODAY'S TWISTER

NAME ________________________________

How many degrees are there between the hands of the clock at 4:40?

ANSWER: ________________________________

125. TODAY'S TWISTER

NAME____________________________________

By how much does the sum of the reciprocals of 2/5 and 2 exceed the reciprocal of their sum?

ANSWER:________________________________

126. TODAY'S TWISTER

NAME____________________________________

Without computing, name all the whole numbers greater than 1 and less than 11 that will not divide evenly into (2 *x* 3 *x* 4 *x* 5 *x* 6 *x* 7 x 8 *x* 9+1).

ANSWER:________________________________

127. TODAY'S TWISTER

NAME____________________________________

If Gretchen is grateful, then Hortence is hopeful. Then which, if any, of the following statements are true?

1. If Gretchen is not grateful, Hortence is not hopeful.
2. If Hortence is hopeful, Gretchen is grateful.
3. If Hortence is not hopeful, Gretchen is not grateful.

.

ANSWER: ________________________________

128. TODAY'S TWISTER

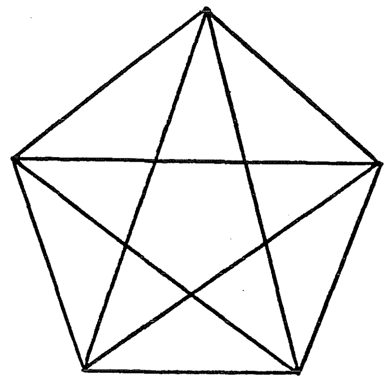
NAME____________________________________

One window is one-half a yard square. Another window has an area of one-half square yard. The area of the first window is what fractional part of the second?

ANSWER: ________________________________

129. TODAY'S TWISTER

NAME ___________________________________

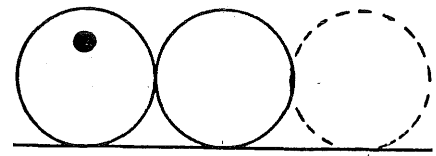


Count the triangles in this figure. Get all of them!

ANSWER: ___________________________

130. TODAY'S TWISTER

NAME ___________________________________



The left hand log has a knot on the end and near the top. That log is rolled over the log in the middle to the position at the right. Show the location of the knot in the right hand position.

ANSWER: ___________________________

131. TODAY'S TWISTER

NAME ___________________________________

$\left[5\frac{2}{3}\right] = 5,\ [3.2] = 3.$ That is, $[n]$ means to take only the greatest integer in n. Thus, $\left[2\frac{2}{3}\right]$ =2, $[100]$ = 100, $[.3]$= 0, $[.8]$= 0

Do these:

a. $\left[2\ \frac{141}{151}\right] + \left[6\ \frac{99}{110}\right]$

b. $\left[2\ \frac{141}{151} + 6\ \frac{99}{110}\right]$

c. $\left[2\ \frac{50}{101} + 3\ \frac{51}{100}\right]$ (Think, don't compute.)

ANSWER: a._______ b. _______ c._______

132. TODAY'S TWISTER

NAME ___________________________________

There are three volumes of books, identical in size and shape, standing next to one another in order. A bookworm begins at the first page of Volume I and eats his way through to the last page of Volume III. If each book is 2 3/4 inches thick with the pages representing 2 1/2 inches of this, how far does the bookworm travel?

ANSWER: ___________________________

133. TODAY'S TWISTER

NAME ______________________________

A man had $1.15 in coins, each less than a dollar, in his pocket. Still, he could not make change for a dollar, a half dollar, a quarter, a dime or a nickel. What coins did he have in his pocket?

ANSWER: ______________________________

134. TODAY'S TWISTER

NAME ______________________________

Mrs. "M" said, "I have no sister and I have no brother, but that girl's mother is daughter to my mother."

Who is "that girl"?

ANSWER: ______________________________

135. TODAY'S TWISTER

NAME ______________________________

Four 4's can be used to express numbers such as:

$2 = \frac{4}{4} + \frac{4}{4}$

$3 = \sqrt{4 + \left(\frac{4}{4}\right) + 4}$

$4 = \sqrt{4} + \sqrt{4} + 4 - 4$

Use four 4's to express 11. Any of the above symbols may used as well as multiplication and decimal points.

ANSWER: ______________________________

136. TODAY'S TWISTER

NAME ______________________________

List the letter of each number which is not the square of some integer:

a. 2501
b. 81
c. $\sqrt{81}$
d. 25.25
e. 90,000,000,000,000
f. 16,000,000
g. 30,147
h. 39 x 3 x 13
i. 6^{15}
j. -100

ANSWER: ______________________________

137. TODAY'S TWISTER

NAME ______________________________

When Penelope broke her bank and counted all her pennies she found that when she counted them two at a time there was one left over. Also, when she counted them three, four, five or six at a time, there was always just one left over. What is the smallest number of pennies that Penelope could have had in her bank if, when she counted them seven at a time, there were none left over?

ANSWER: ______________________________

138. TODAY'S TWISTER

NAME ______________________________

Show using a drawing how six toothpicks may be glued together to form four equilateral triangles which are congruent (same size and shape).

ANSWER: ______________________________

139. TODAY'S TWISTER

NAME ______________________________

Recall that [n] means the largest whole number in n: [8.7] = 8, $\left[\frac{5}{8}\right]$ =0 etc. What is the smallest whole number which can be put in both boxes so this sentence will still work?

$$\left[\frac{\square}{3}\right] = \left[\frac{\square}{4}\right]$$

ANSWER: ______________________________

140. TODAY'S TWISTER

NAME ______________________________

How long would it take to cut a 387 inch length of string **into** 9 inch lengths if each cut takes two seconds? Only one string is cut at a time.

ANSWER: ______________________________

141. TODAY'S TWISTER

NAME ____________________________________

Two cyclists start at the same time from the same place travel to a common finish point at the other side of a hill. Tom travels over the hill, averaging 3 miles per hour up for the first half and 11 miles per hour down for his second half. Sue goes around the hill at a constant 7 miles per hour. Which of these is correct if their total distances are equal?

a. Tom beats Sue
b. Sue beats Tom
c. They tie
d. Not enough information to tell

ANSWER: ____________________________

142. TODAY'S TWISTER

NAME ____________________________________

Name five numbers less than 20 which, when squared, give perfect cubes. Each number contains one digit.

ANSWER: ____________________________

143. TODAY'S TWISTER

NAME ____________________________________

Which of the following is (are) evenly divisible by 2, 3, 6 and 9 (by all four numbers)?

156	236	5,364
4,010	1,010	207

ANSWER: ____________________________

144. TODAY'S TWISTER

NAME ____________________________________

True or false: The statement "4/.03 is not greater than 3/.03" is not false.

ANSWER: ____________________________

145. TODAY'S TWISTER

NAME ____________________________________

Four boys have three pizzas to share. The pizzas have diameters of nine, twelve, and fifteen inches. Can you describe a way to divide the pizzas fairly using only 2 cuts?

ANSWER: ____________________________________

146. TODAY'S TWISTER

NAME ____________________________________

A girl went to a booth in an amusement park and said to the proprietor, "If you give me as much money as I now have then I will spend $10.00 at your booth. It was done and repeated at a second and third booth. How much money did the girl have when she started at the first booth if she finished with no money left?

ANSWER: ____________________________________

147. TODAY'S TWISTER

NAME ____________________________________

Back when $2 bills were in circulation a man bought a watch for $103 including tax. Being rather an unusual person he paid for it in eight bills. How could this have been done if:

a) He used any bills except $2 bills

b) He used any bills except $1 bills

ANSWER: a. ______________________________

b. ______________________________

148. TODAY'S TWISTER

NAME ____________________________________

A conversation heard on a bus:

Man: “Was he related to you?"
Woman: "Yes. That gentleman's mother was my mother's mother-in-law, but he was not on speaking terms with my father."
Man: "Of course."

But you could see that he was not a lot wiser. How was the gentleman who was referred to by the woman related to the woman?

ANSWER: ____________________________________

149. TODAY'S TWISTER

NAME ____________________________________

The greatest integer in – 3.2 is – 4.
Think about it.

So [-63.8] = ______, [0] = _____, [-1/2] = _______.

150. TODAY'S TWISTER

NAME ____________________________________

Nicotine Nelly walked along the street picking up cigarette butts and used the tobacco in them to roll her own. If a standard size cigarette can be rolled out of six standard size butts, how many cigarettes can rolled and smoked if Nellie finds 36 standard size butts?

ANSWER: ______________________________

151. TODAY'S TWISTER

NAME ____________________________________

Two trains start out at 7:00 A.M. One travels from Slobovia to Berlin and the other from Berlin to Slobovia. The first train makes the trip in 8 hours and the second in 12 hours. At what time of day will the two trains pass each other?

ANSWER: ______________________________

152. TODAY'S TWISTER

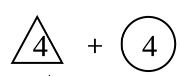
NAME ____________________________________

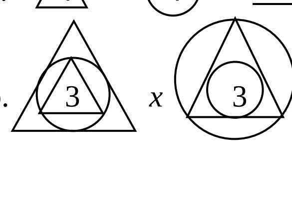
If a number appears in a triangle, it means to double the number and add 3.

Thus,  = 11

And suppose that  means to find the reciprocal of the number.

Find the value of:

a.  = __________

b.  = __________

153. TODAY'S TWISTER

NAME ______________________________________

A certain prime number between 100 and 200 is the same when written backwards but when written upside down and looked at in the mirror, is divisible by 7. What is it?

ANSWER: ______________________________

154. TODAY'S TWISTER

NAME ______________________________________

Mr. Tazzlewourtz hired a boy at a certain hourly wage, reduced his wage by ten percent, and later raised it by ten percent. The boy's new wage was then 2¢ less than the old. What was the new hourly wage?

ANSWER: ______________________________

155. TODAY'S TWISTER

NAME ______________________________________

Twice a fraction plus half that fraction times that fraction equals that fraction. What's the fraction?

ANSWER: ______________________________

156. TODAY'S TWISTER

NAME ______________________________________

A is less than B. If C is less than B then D is greater than E. What single definite conclusion can you come to regarding A if D = 5 and E = 6 ?

ANSWER: ______________________________

157. TODAY'S TWISTER

NAME ___________________________________

On a chessboard of 64 squares a man agreed to put a. penny for his son on the first square the first day, 2¢ on the second square the second day, 4¢ on the third square the third day, then 8¢, 16¢, etc. How much money he have given his son after 64 days?

a. About $100
b. About $60,000
c. About 90 million dollars
d. About 18 quintillion dollars

ANSWER: ___________________________________

158. TODAY'S TWISTER

NAME ___________________________________

There are less than six dozen eggs in a basket. If I count them two at a time, there is one left. If I count them three at a time there are two left. If I count them four at a time there are three left. And if I count them five at a time there are four left. How many eggs are there in the basket?

ANSWER: ___________________________________

159. TODAY'S TWISTER

NAME ___________________________________

$$\frac{1}{2} + \frac{1}{5} + \frac{1}{7} = \frac{47}{70}$$

Find a set of having different denominators but all having numerators of 1 so that the fractions add up to $\frac{17}{31}$. Hint: $\frac{1}{2} = \frac{15\frac{1}{2}}{31}$

ANSWER: ___________________________________

160. TODAY'S TWISTER

NAME ___________________________________

Between ten and twenty there are two consecutive integers which, when squared, have the same digits in different order. What are they?

ANSWER: ___________________________________

161. TODAY'S TWISTER

NAME______________________________________

Two mothers and two daughters left town. This resulted in a reduction in the population of' three, How could this be?

ANSWER: ______________________________

162. TODAY'S TWISTER

NAME______________________________________

Halenthorpe rides his bike 5 miles to his grandmother's house' at the rate of 10 miles per hour. At what rate should he return home so that the average rate for the whole trip will be 12 miles per hour?

ANSWER:________________________________

163. TODAY'S TWISTER

NAME______________________________________

Compute in simplest form the reciprocal of the sum of the reciprocals of 0.5 and 3.

ANSWER:________________________________

164. TODAY'S TWISTER

NAME______________________________________

(Dirty trick problem)

Give the next three terms of this series:

O,T, T, F, ____, ____, ____

ANSWER: ______________________________

165. TODAY'S TWISTER

NAME ____

If a question is not fair, say so. If it is fair, give the answer. Recall that [n] means the largest integer in n.

a. $[(0.9)^{100}]$ = ______

b. $[0.9]^{100}$ = ______

c. $[(1.9)^{100}]$ = ______

d. $[1.9]^{100}$ = ______

166. TODAY'S TWISTER

NAME ____

[373/374 + 374/373] = ___________

ANSWER: ________________________________

167. TODAY'S TWISTER

NAME ____

How many degrees are there between the hands of the clock at 12:01?

ANSWER: ________________________________

168. TODAY'S TWISTER

NAME ____

A can do a piece of work in 7 days. B is 50% more efficient than A. How many days will it take B to do the same piece of work?

ANSWER: ________________________________

169. TODAY'S TWISTER

NAME ________________________________

As the denominator of a fraction increases, the value of the fraction decreases: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc.

What value does this complex fraction get closer and closer to as the number represented by y increases?

$$\frac{1}{1-\frac{1}{y}}$$

ANSWER: ________________________________

170. TODAY'S TWISTER

NAME ________________________________

What perfect square numbers having two digits give prime numbers when decreased by two? Name all of them.

ANSWER: ________________________________

171. TODAY'S TWISTER

NAME ________________________________

Three quickies; get them all.

1. Four years ago the sum of the ages of two children was 11 years. What is their sum now? ________
2. Bill can wax a car in 3 hours and Tom can wax it in 6. Working together they can wax it in _____hours. (The answer is one of these: 2 hours, $4\frac{1}{2}$ hours, 9 hours.)
3. Atty Timwater had seventeen white mice and all but five died. How many did he have left? _________

172. TODAY'S TWISTER

NAME ________________________________

Two math classes took the same test. One class of 20 students had an average grade of 80%. The other class of 30 students had an average grade of 70%. What was the average grade of all students in both classes?

ANSWER: ________________________________

173. TODAY'S TWISTER

NAME ______________________________________

A man was buying a certain item at the store. The clerk said: "One costs ten cents. Seven will cost ten cents. Eleven will cost 20¢."

What was the man buying?

ANSWER: ______________________________

174. TODAY'S TWISTER

NAME ______________________________________

On the number line a (+6) tadhopper starts on 1 and jumps forward, to 7, 13, 19, etc. Also, a (-4) tadhopper starts on 109, hops downward to 105, 101, 97, etc. Name all the numbers they both touch in common (not necessarily at the same time) between 45 and 89.

ANSWER: ______________________________

175. TODAY'S TWISTER

NAME ______________________________________

Compute in simplest form the reciprocal of the sum of the reciprocals of 0.5 and 3.

ANSWER: ______________________________

176. TODAY'S TWISTER

NAME ______________________________________

AAAA
AAAB
AABA
AABB
ABAA
ABAB
ABBA
ABBB

The column to the left contains patterns which may be used to discover the next two four lettered members. Try it.

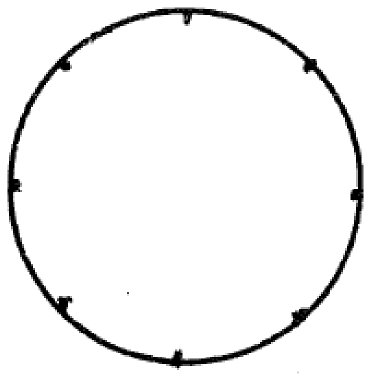
____
____

ANSWERS

177. TODAY'S TWISTER

NAME ______________________________

Eight points are spaced equally around a circle. How many different straight lines can you draw so that each connects two of the point??



ANSWER: ______________________________

178. TODAY'S TWISTER

NAME ______________________________

Give the letter of each expression which has a value of 1 – 10 inclusive.

a. $\sqrt{1\frac{1}{3}}$

b. $9 \; x \; 0.\overline{11}$ (repeating decimal)

c. The sum of the three smallest factors of 1925

d. $(10)^0$

e. The sum of 35 terms of:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+etc.$$

f. $(0.9)^{50}$

ANSWERS: ______________________________

179. TODAY'S TWISTER

NAME ______________________________

Find the two numbers whose product is 512 and whose quotient is 2.

ANSWER: ______________________________

180. TODAY'S TWISTER

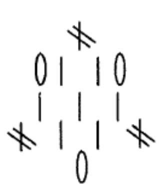
NAME ______________________________

List the letter of each which always has an even number as an answer.

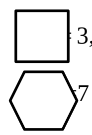
a. The sum of three consecutive whole numbers
b. The product of three consecutive whole numbers
c. The sum of four consecutive whole numbers
d. The sum of five consecutive whole numbers

ANSWER: ______________________________

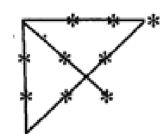
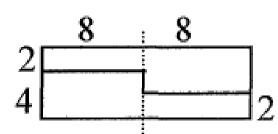
Twister Answers and Some Solution Comments

Twister 1	Twister 2	Twister 3	Twister 4
7 24 586 2 Or any of its rotations	24	91 and 130	0
Twister 5	Twister 6	Twister 7	Twister 8
Jack of spades, queen of spades, queen of hearts.	Fill 5, dump into 3, empty that 3. Dump the remaining 2 from the 5 into the 3. Fill the 5, dump into 3's space (1). 4 are now left in the 5! You can also start with 3 & get 5.	$3	8a. $6\frac{6}{6}$ 8b. $\sqrt{4} + \sqrt{4} + \frac{4}{4}$ Other solutions possible.
Twister 9	Twister 10	Twister 11	Twister 12
123455678900 Curve ball question: ones should be added diagonally to stay in proper place value columns. 10987654321 is not correct.	\10 $\frac{7}{12}$	28	64
Twister 13	Twister 14	Twister 15	Twister 16
30	4	$\frac{21}{110}$	Move the nine from the last group to the first.
Twister 17	Twister 18	Twister 19	Twister 20
284 Divisors are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110	Harry $5, Cary $7	White. This happens only with the North Pole as a starting point. Geometrically, it could also start 3+ miles from the South Pole but there are no bears there.	80 cents
Twister 21	Twister 22	Twister 23	Twister 24
Build it on the North Pole	4½ and 1½	345676543 (no perfect square can end in 3	38 days
Twister 25	Twister 26	Twister 27	Twister 28
a. 2 b. 29 c. 53 d. 97	Many possible answers.	O and R (not N and O)	4 feet
Twister 29	Twister 30	Twister 31	Twister 32
	Double and add 1, double and subtract one, etc. Answer, 139	124, 86, 11, 9, 3 (not 6), 5.	L=8, M = 9, N = 1

Twister Answers and Some Solution Comments

Twister 33	Twister 34	Twister 35	Twister 36
9 ½ miles per day	28	1/48	$19\frac{1}{5}$ sq. in.
Twister 37	**Twister 38**	**Twister 39**	**Twister 40**
15 cents	156	* **** **** * or **** * * ****	3
Twister 41	**Twister 42**	**Twister 43**	**Twister 44**
$45	Base six	$13\frac{1}{5}$ seconds ($1\frac{1}{5}$ seconds per interval)	a
Twister 45	**Twister 46**	**Twister 47**	**Twister 48**
$7\frac{1}{2}$	a. 23, 28, 33 b. 15, 21, 28 c. 63, 127, 255 d. 55, 91, 140 (Add squares 4, 9,...)	 3, 7	56
Twister 49	**Twister 50**	**Twister 51**	**Twister 52**
6 (Note: This is the inverse idea of #34)	b. This would work for the moon or a basketball, and it could be verified for them.	91	14, 10
Twister 53	**Twister 54**	**Twister 55**	**Twister 56**
The letter “o”	6 hours	45	-2, 97, 8, 4, (Square and subtract 3)
Twister 57	**Twister 58**	**Twister 59**	**Twister 60**
20 miles. (It takes the cyclists one hour to meet)	a. Sally is sad b. No conclusion c. No conclusion d. Harry is not happy	60ft. (6 2/3 ft. per interval)	12 + 3 + 4 + 56 + 7 + 8 + 9
Twister 61	**Twister 62**	**Twister 63**	**Twister 64**
$33 - 3 + \frac{3}{3}$, $3^3 + 3 + \frac{3}{3}$ others possible	4 – 17’s & 2 – 16's	a. 6 b. 9	d, f, h

Twister Answers and Some Solution Comments

Twister 65	Twister 66	Twister 67	Twister 68
14	13 – It is easy to forget those busses enroute as he leaves plus those that depart while he is enroute.	Neither – both distances are equal.	22 $\frac{1}{4}$ of the area of the large circle, since the small semicircles' radii are $\frac{1}{2}$ as large.
Twister 69	**Twister 70**	**Twister 71**	**Twister 72**
20 hours. An $\frac{1}{4}$ increase gives a 5:4 speed ratio. Time ratio is inverse, or 4:5, hence 20 hours.	$3^3 + 3$	120	$\frac{4}{5}$ doz., 9.6 eggs
Twister 73	**Twister 74**	**Twister 75**	**Twister 76**
4 hours	$\frac{1}{2}$ or 50-50. Even + odd reminder not in problem.	26	$77 + \frac{7}{0.7}$ (others?)
Twister 77	**Twister 78**	**Twister 79**	**Twister 80**
$\frac{5}{7}$	All true, though many won't agree, (e) follows from (a-d).	$3\frac{3}{4}$ days. One boy is equivalent to $\frac{2}{5}$ man so, "$3\frac{1}{5}$ men" are working, needing $\frac{3}{3\frac{1}{5}}$ of 4 days.	6 units
Twister 81	**Twister 82**	**Twister 83**	**Twister 84**
3	28 1 goose lays 1 egg in $1\frac{1}{2}$ days; in 6 days (4 times as long), 4 x 7 = 28 eggs laid.	 or a rotation of this	$\frac{9}{58}$
Twister 85	**Twister 86**	**Twister 87**	**Twister 88**
Exchange horses	No number; 2 results in division by 0.	30 days	
Twister 89	**Twister 90**	**Twister 91**	**Twister 92**
	46	a. 40,320 b. 9,506	0, 1, 64
Twister 93	**Twister 94**	**Twister 95**	**Twister 96**
44 (inches)	a. 64 b. 512	a. $4095 b. $4096 c. $8191	110 mph

Twister Answers and Some Solution Comments			
Twister 97	Twister 98	Twister 99	Twister 100
11 inches	a. 0 b. 0 c. 0 d. 8 e. 12 f. 6 g. 1	c > a	$2\frac{19}{26}$
Twister 101	Twister 102	Twister 103	Twister 104
24	18	$\frac{2}{15}$	The big Indian was his mother.
Twister 105	Twister 106	Twister 107	Twister 108
19 days	3	10	11, 31, 41, 61, 71
Twister 109	Twister 110	Twister 111	Twister 112
12 hours 1 boy with 1 car in 6 hours, so 3 boys with 6 cars in 12 hours	16 lbs.	5,050 Add the rows; there are 100 101's, twice the sum (two rows); 10100/2 =5050	$1.20
Twister 113	Twister 114	Twister 115	Twister 116
2 5 2 5 5 2 5 2	8913 Note: four thousand fourteen hundred = 4000+1400=5400	11 cents	0 and 1
Twister 117	Twister 118	Twister 119	Twister 120
-- 1, 2	816 357 492 Or one of its rotations	50 cents & 75 cents	7 seconds for 6 time intervals $=1\frac{1}{6}$ seconds per interval; 10 strikes take 9 intervals; $9\ x\ 1\frac{1}{6} = 10\frac{1}{2}$
Twister 121	Twister 122	Twister 123	Twister 124
888 + 88 + 8 + 8 + 8 (Other solutions?)	136 822 ÷ 6 = 137; 136 'separators'	7	100 degrees The hour hand moved 2/3 of the 30 degrees between 4 & 5
Twister 125	Twister 126	Twister 127	Twister 128
$2\frac{7}{12}$	2 through 10. "The product" can share no factors with "the product plus 1". (No two consecutive numbers can have factors in common except the factor 1.) 11 is the first possibility.	#3 #1 and #2 are often implicit in speech but do not follow logically.	$\frac{1}{2}$ ($2\frac{1}{4}$ sq. ft. to $4\frac{1}{2}$ sq. ft.)

Twister Answers and Some Solution Comments

Twister 129	Twister 130	Twister 131	Twister 132
35 There are 7 kinds of triangles, 5 of each.		a. 8 b. 9 c. 6. Note the difficult reasoning needed in part c. $\frac{51}{100}$ exceeds $\frac{1}{2}$ by more than $\frac{50}{101}$ falls short of $\frac{1}{2}$.	3"
Twister 133	**Twister 134**	**Twister 135**	**Twister 136**
Half dollar, a quarter, and 4 dimes.	Mrs. M's daughter.	11 = 4/0.4 + 4/4; others (?).	a, d, e, g, i, j. e has odd number of zeros. g ends in 7. i has odd number of 6's multiplied. (10000 is square of an integer,1000 is not.)
Twister 137	**Twister 138**	**Twister 139**	**Twister 140**
301	A tetrahedron	3	84 seconds
Twister 141	**Twister 142**	**Twister 143**	**Twister 144**
b Tom spends more of his time at his <u>slower</u> rate, so his average rate is less than it would be on the level. So b, Sue wins.	-8, -1, 0, 1, 8	5364 (Use divisibility rules.)	False
Twister 145	**Twister 146**	**Twister 147**	**Twister 148**
The sum of the two smallest pie areas equals the largest so cut each in half. 1 big half = a medium half plus a small half. Note: this could be done with 1 cut if pies are stacked concentrically	$8.75	a. 50, 20, 20, 5, 5, 1, 1, 1 b. 50, 20, 20, 5, 2, 2, 2, 2	Her Uncle
Twister 149	**Twister 150**	**Twister 151**	**Twister 152**
-64, 0 and -1	7 Don't forget the cigarette accumulated from the butts her smoking creates!	11:48 A.M. , a toughie. The time ratio is 2:3 (5 parts). The fast train goes $\frac{3}{5}$ of its total time while the slower train goes $\frac{2}{5}$ of its total time. Either way, $\frac{3}{5} x\ 8 = \frac{2}{5} x\ 12 = 4\frac{4}{5}$ hours (after 7 A.M.)	a. $11\frac{1}{4}$ b. $\frac{29}{33}$
Twister 153	**Twister 154**	**Twister 155**	**Twister 156**
191	$1.98	$\frac{2}{5}$	a < b

Twister Answers and Some Solution Comments			
Twister 157	Twister 158	Twister 159	Twister 160
d	59	$\frac{1}{2}+\frac{1}{31}+\frac{1}{62}$	13 and 14
Twister 161	Twister 162	Twister 163	Twister 164
Grandmother, mother and daughter	15 mph (Hard!) $\frac{10\ (distance)}{12}=\frac{5}{6}$ hr. $\frac{1}{2}$ hr. to go, so $\frac{1}{3}$ hr. to return. $\frac{5\ (return\ distance)}{\frac{1}{3}}=15$	3/7	F,S,S-first letters of Five, Six, Seven
Twister 165	Twister 166	Twister 167	Twister 168
a. 0 b. 0 c. not fair d. 1	2. Note that the first fraction is 1/374 below 1, but the second is 1/373 above 1. Which one wins?	$5\frac{1}{2}$ degrees The minute hand goes 6 degrees of clock face arc and the hour hand goes $\frac{1}{12}$ as far, or $\frac{1}{2}$ degree. $6-\frac{1}{2}=5\frac{1}{2}$	$4\frac{2}{3}$ days 50% more means $\frac{3}{2}$ times, taking $\frac{2}{3}$ of A's time.
Twister 169	Twister 170	Twister 171	Twister 172
1	25, 49, 81	19, 2, 5	74%
Twister 173	Twister 174	Twister 175	Twister 176
House numerals	49, 61, 73, 85	3/7	BAAA, BAAB
Twister 177	Twister 178	Twister 179	Twister 180
28	a, b, d	32, 16	b, c